ALGEBRA QUALIFYING EXAM SPRING 2018

Exercise 1. Suppose p is a prime. Show that the Galois group of $x^5 - 1 \in \mathbb{F}_p[x]$ depends only on $p \pmod 5$, and compute it for each congruence class of $p \pmod 5$.

Exercise 2. Let R be a Dedekind domain with field of fractions K Show that for any two proper fractional ideals I, J there are $\alpha, \beta \in K$ with $\alpha I, \beta J \subseteq R$ integral and $\alpha I + \beta J = R$.

Exercise 3. Suppose that R is a Noetherian ring and $\mathfrak{p} \subseteq R$ is a prime ideal such that $R_{\mathfrak{p}}$ is an integral domain. Show that there is an $f \in R \setminus \mathfrak{p}$ such that R_f is an integral domain where $R_f = S^{-1}R$ with $S = \{1, f, f^2, f^3, \ldots\}$.

Exercise 4. Let k be an algebraically closed field. Consider the affine variety $V = k^2$ (with coordinates x, y), and the affine variety $W = k^2$ (with coordinates s, t). Suppose $\varphi : V \to W$ is a morphism, and denote by $R \subseteq k[x, y]$ the image of the induced ring homomorphism $\tilde{\varphi} : k[s, t] \to k[x, y]$. For each of the following statements, give a proof or a counterexample.

(1) If φ has Zariski dense image, then φ is surjective.
(2) If $k[x, y]/R$ is an integral extension of rings, then φ is surjective.