Two-Sided Matching via Balanced Exchange *

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Abstract

We introduce a new matching model to mimic two-sided exchange programs such as tuition and worker exchanges, in which export-import balances are required for longevity of programs. These exchanges use decentralized markets, making it difficult to achieve this goal. We introduce the *two-sided top-trading-cycles*, the unique mechanism that is balanced-efficient, worker-strategy-proof, acceptable, individually rational, and respecting priority bylaws regarding worker eligibility. Moreover, it encourages exchange, because full participation induces a dominant-strategy equilibrium for firms. We extend it to dynamic settings permitting tolerable yearly imbalances and demonstrate that its regular and tolerable versions perform considerably better than models of current practice.

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1 Introduction

We introduce and model a new class of two-sided matching markets without explicit transfers, in which there is an additional fundamental constraint.[1] The eventual market outcome is linked to an initial status-quo matching, which may give participants certain rights that constrain how future activity can play out. Since market outcome is typically different from the status quo, such activities loosely resemble an *exchange* in which one side of the agents are changing or acquiring new partners in addition to the two-sided matching market structure. In such markets, a fundamental *balancedness* condition needs to be sustained with respect to the status-quo matching. The motivation for such a balancedness constraint can be different depending on the features of the market. Two contrasting examples are labor and higher education markets, where workers and colleges provide services to be compensated, respectively. In *worker exchange*, a worker needs to be replaced with a new one at her home firm so that this firm can function properly, and thus, the market needs to clear in a balanced manner. In *student exchange*, the college that is matched with an exchange student should be able to send out a student as well so that its education costs do not increase, and thus, the market needs to clear in a balanced manner. There are several prominent examples of such exchanges, such as national and international teacher-exchange programs, clinical-exchange programs for medical doctors, worker-exchange programs within or across firms, and student-exchange programs among colleges. This balancedness constraint induces preferences for firms/colleges not only over whom they get matched with (i.e., *import*), but also over whom they send out (i.e., *export*). The most basic kind of such preferences requires the firm/college to have a preference for balanced matchings, i.e., for import and export numbers to be equal. We analyze our model over two explicit market applications: (permanent) tuition exchange and temporary worker exchange (see Section 2 for details).

In *tuition exchange*, the two sides are colleges and students. Each student who is a dependent of a faculty member at a college can attend another institution for free, if admitted as part of a tuition-exchange program. (Tuition exchange is a part of the benefit plan of the faculty member.) Colleges have preferences over matchings. We assume only a weak structure for these preferences. Colleges' rankings over the incoming class is assumed to be responsive to their strict rankings over individual students. Moreover,

[1]The theory and design of two-sided matching markets, such as entry-level labor markets for young profxessionals, online-dating markets, or college admissions, have been one of the cornerstones of market design for more than thirty years (see Gale and Shapley, 1962; Roth, 1984; Roth and Peranson, 1999; Hitsch, Hortaçsu, and Ariely, 2010). Moreover, the theory of these markets has some important applications in allocation problems such as student placement and school choice (see Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003).

their preferences over matchings are determined through their rankings over the incoming class and how balanced the eventual matching is.[2] We start by showing, through a simple example, that individual rationality and nonwastefulness, standard concepts in two-sided matching markets, and balancedness are in general conflicting requirements (Proposition 1). For this reason, we restrict our attention to the set of balanced-efficient mechanisms. Unfortunately, there exists no balanced-efficient and individually rational mechanism that is immune to preference manipulation for colleges (Theorem 2).

We propose a new two-sided matching mechanism that is balanced-efficient, student group strategy-proof, acceptable, respecting internal priorities,[3] individually rational, and immune to quota manipulation by colleges (Theorems 1, 3, and 4). We also show that it is the unique mechanism satisfying the first four properties (Theorem 5). To our knowledge, this is one of the first papers using axiomatic characterization in the context of practical market design.

The outcome of this mechanism can be computed with a variant of David Gale's top-trading-cycles (TTC) algorithm (Shapley and Scarf, 1974). In the school choice problem (Abdulkadiroğlu and Sönmez, 2003) and the house allocation problem with existing agents (Abdulkadiroğlu and Sönmez, 1999), variants related to Gale's TTC, have been introduced and their properties have been extensively discussed (also see Pápai, 2000). In all of these problems, one side of the market is considered to be objects to be consumed that are not included in the welfare analysis. Moreover, they are not strategic agents. In two-sided matching via exchange, in contrast to school choice and house allocation, both sides of the market are strategic agents and must be included in the welfare analysis. Based on these variants of Gale's TTC, we formulate our algorithm, and thus, we refer to the induced mechanism as the *two-sided top trading cycles* (2S-TTC). As far as we know, this is the first time a TTC-variant algorithm has been used to find the outcome of a two-sided matching mechanism.[4]

[2]We do not rule out colleges having more complex preferences over which students they send out.

[3]A mechanism *respects internal priorities* if, after a college increases the number of sponsored students, every student who was initially sponsored by that college is not hurt.

[4]Ma (1994) had previously characterized the core of a house exchange market, which can be found by Gale's TTC algorithm, when each house has a unit quota through Pareto efficiency, individual rationality, and strategy-proofness for students. Our characterization uses a proof technique different not only from Ma (1994), but also subsequent simpler proofs of this prior result by Sönmez (1995) and Svensson (1999). There are a few other TTC-related characterization results in the literature: Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2017); Dur (2012); Morrill (2013) characterize school choice TTC a la Abdulkadiroğlu and Sönmez (2003); Pycia and Ünver (2017) characterize general individually rational TTC rules a la Pápai (2000) when there are more objects than agents; and Sönmez and Ünver (2010) characterize TTC rules a la Abdulkadiroğlu and Sönmez (1999) for house allocation with existing tenants. Kesten (2006) provides the necessary structure on the priority order to guarantee fairness of school choice TTC. Besides these characterizations, a related mechanism to ours was proposed by Ekici (2011) in an

Although 2S-TTC is balanced-efficient, it may not match the maximum possible number of students while maintaining balance. We show that if the maximal-balanced solution is different from the 2S-TTC outcome for some preference profile, it can be manipulated by students (Theorem 6).

Some tuition exchange programs require keeping a balance in a moving three-year window for their member colleges. For this reason, we extend our model to a dynamic setting, where colleges can have tolerable yearly imbalances. We propose an extension, *two-sided tolerable top-trading-cycle* (2S-TTTC) mechanism, which allows one to keep the imbalance of each college between some upper and lower bounds, and these bounds can be adjusted over the years. Once the bound-setting and adjustment processes are externally set, we show that 2S-TTTC keeps good properties of 2S-TTC: it is student-strategy-proof, acceptable, and respecting internal priorities; moreover, no acceptable matching within the balance limits can Pareto dominate its outcome.[5]

As the last part of our analysis of tuition-exchange programs, in Online Appendix I, we compare the performances of 2S-TTC and 2S-TTTC with that of the best-case scenario of the current practice of tuition exchange with a wide range of simulations.[6] By considering different degrees of correlation among students' and also colleges' preferences, and different yearly imbalance tolerance levels, we show that 2S-TTC and its variant match considerably more students to colleges and increase students' welfare over the naive student-proposing deferred acceptance outcome, the best case scenario for the current market.[7] This is a best-case scenario for the decentralized market as it minimizes coordination failures and ignores possible college incentives to underreport their certification quotas.[8] Moreover, Combe, Tercieux, and Terrier (2016) conducted an empirical study using teacher assignment data from France using a model related to ours.

object allocation problem for temporary house exchanges with unit quotas.

[5]The closest in the literature to 2S-TTTC's algorithm is the top-trading-cycles-and-chains (TTCC) algorithm proposed by Roth, Sönmez, and Ünver (2004); however, the use and facilitations of "chains" are substantially different in this algorithm than in 2S-TTTC.

[6]We also develop a model of current semi-decentralized practice in tuition exchange in Online Appendix A. We show that balancedness is not in general achieved through decentralized market outcomes, jeopardizing the continuation and success of such markets. We define stability for particular externalities in college preferences. We show that stable matchings exist when colleges have plausible preferences over matchings (Proposition 3). Moreover, Proposition 4 implies that stability and balancedness are incompatible. Then we show that stability discourages exchange and can prevent the market from extracting the highest gains from exchange (see Theorems 10 and 11).

[7]It is the best case, since in the Online Appendix A, we show that under reasonable assumptions, the current market gives incentives to colleges to decrease their quotas.

[8]It should be noted that there could be other market structures not governed by our simulation generating distributions such that the results we find do not hold. Thus, these simulations are examples of domains in which 2S-TTC or its variant dominate the best case outcomes of decentralized markets under a vast majority of parameters.

Compared to deferred-acceptance-based current practice, they show that a TTC-based approach doubles the number of teachers moving from their initial assignment. Additionally, when the distribution of the ranks of teachers over the schools are considered, the outcome of the TTC-based approach stochastically dominates that of the current practice. Thus, there exist real-life settings, in which our proposals can lead to significant welfare improvements.

We extend this model for temporary worker exchanges, such as teacher-exchange programs. We tweak our model slightly and assume that the quotas of the firms are fixed at the number of their current employees, and, hence, firms would like to replace each agent who leaves. We also assume that firm preferences are coarser than colleges in tuition exchange due to the temporary nature of the exchanges. We assume they have *weakly size-monotonic* preferences over workers: larger groups of acceptable workers are weakly better than weakly smaller groups of acceptable workers when the balance of the matching with larger groups of acceptable workers is zero and the balance of the matching with smaller group of worker is nonpositive.[9] In this model, we prove that 2S-TTC not only carries all of its previous properties through but also is strategy-proof for the firms, making it a very viable candidate (Theorem 9). Our aforementioned characterization also holds in this model.

2 Applications

2.1 Tuition Exchange

Some of the best-documented matching markets with a balancedness requirement are *tuition-exchange programs* in the US. These are semi-decentralized markets, and some have failed over the years because of problems related to unbalanced matching activity.

It has been difficult for small colleges and universities to compete with bigger schools in trying to hire the best and brightest faculty. Colleges located farther away from major metropolitan areas face a similar challenge. Tuition-exchange programs play a prominent role for these colleges in attracting and retaining highly qualified faculty.[10]

[9] Weakly size-monotonic preferences are weaker than dichotomous preferences (in absence of externalities), which are widely used in the matching literature, see for example Bogolomania and Moulin (2004), Roth, Sönmez, and Ünver (2005), Roth, Sönmez, and Ünver (2007), Ekici (2011), and Sönmez and Ünver (2014).

[10] "Tuition Exchange enables us to compete with the many larger institutions in our area for talented faculty and staff. The generous awards help us attract and retain employees, especially in high-demand fields like nursing and IT." – Frank Greco, Director of Human Resources, Chatham University, from the home page of The Tuition Exchange, Inc., www.tuitionexchange.org, retrieved on 09/19/2012. Also see Online Appendix C about the results of a survey that we conducted detailing the importance of

Many colleges give qualified dependents of faculty tuition waivers. Through a tuition-exchange program, they can use these waivers at other colleges and attend these colleges for free. The dependent must be admitted to the other college. Tuition exchange has become a desirable benefit that adds value to an attractive employment package without creating additional out-of-pocket expenses for colleges; that is, colleges do not transfer money to each other for accepting their faculty's dependents.

One of the prominent programs is "The Tuition Exchange, Inc." (TTEI), which is also the oldest and largest of its kind.[11]

Each participating college to TTEI establishes its own policies and procedures for determining the eligibility of dependents for exchange and the number of scholarships it will grant each year. Each member college has agreed to maintain a balance between the number of students sponsored by that institution ("exports") and the number of scholarships awarded to students sponsored by other member colleges ("imports"). Colleges aim to maintain a one-to-one balance between the number of exports and imports. In particular, if the number of exports exceeds the number of imports, then that college may be suspended from the tuition-exchange program.[12] Colleges often set the maximum number of sponsored students in a precautionary manner. Many colleges explicitly mention in their application documents that in order to guarantee their continuation in the program, they need to limit the number of sponsored students.[13] As a result, in many cases not all qualified dependents are sponsored.

A tuition-exchange program usually functions as follows: each college determines its quotas, which are the maximum number of students it will sponsor (its "eligibility quota") and the maximum number it will admit (its "import quota") through the program. Then, the eligible students apply to colleges, and colleges make scholarship decisions based on preferences and quotas. A student can get multiple offers. She declines all but one, and, if possible, further scholarship offers are made in a few additional rounds. Students who are not sponsored cannot participate in the program, and hence do not receive a tuition-exchange scholarship. The admitting institution de facto awards a tuition waiver to the dependent of the faculty of another college.

tuition-exchange programs in job choice for faculty members.

[11]See http://www.tuitionexchange.org. Through TTEI, 7,000 scholarships were awarded in 2015-2016, with annual value of $34,000 per scholarship that is paid as a tuition reduction. Despite TTEI's large volume, other tuition exchange programs clear a significant number of all exchange transactions in the US. In Online Appendix C, we describe the features of prominent tuition exchange programs.

[12]See http://www.sxu.edu/admissions/financial_aid/exchange.asp reached on Feb 19, 2018.

[13]Lafayette College, Daemen College, DePaul University, and Lewis University are just a few examples.

2.2 Temporary Worker Exchanges

The balancedness requirement also matters in *temporary worker-exchange programs* (such as those for teachers, students, academic staff, and medical doctors). The Commonwealth Teacher Exchange Programme (CTEP), Fulbright Teacher Exchange Program, Erasmus Student Exchange Program, and the exchange program of International Federation of Medical Students' Associations (IFMSA) are just a few examples.[14] Some of these have been running for decades,[15] and thousands of participants benefit from these worker exchange programs annually. Every year more than 10,000 medical students and 200,000 college students around the world participate in IFMSA's and Erasmus' exchanges, respectively.[16] The main difference between these programs and tuition exchange is that (1) most exchange appointments are temporary, typically lasting one year, and (2) the workers are currently employed by their associated firms, so if they cannot be exchanged, they will continue to work in their current jobs.

2.3 Importance of Balancedness Requirement

Although two-sided matching via exchange induces a two-sided matching market, workers (students) cannot participate in market activity unless their home firms (colleges) sponsor them. Hence, an import/export balance emerges as an important feature of sustainable outcomes, as there are no monetary transfers between parties and there are costs for colleges associated with providing students. Balance requirements are the most important feature of these markets that distinguishes them from the previously studied matching markets. We illustrate three cases in which the absence of a balanced exchange led to the failure of the exchange program in different contexts.[17]

The Northwest Independent Colleges Tuition Exchange program was founded in 1982 and included five members. Unlike TTEI, the colleges were not able to limit their exports. Because of sizable imbalances between imports and exports, members agreed to dissolve the program, and it stopped accepting new applicants after Fall 2015 in its cur-

[14] There are also small bilateral staff-exchange programs. See Online Appendix D for details.

[15] CTEP, which allows participants to exchange teaching positions and homes with a colleague from the UK, Australia, or Canada, has been running for 100 years. See http://www.cyec.org.uk/exchanges/commonwealth-teacher-exchange reached on Feb 19, 2018.

[16] See http://www.amsa.org/AMSA/Homepage/EducationCareerDevelopment/InternationalExchanges.aspx and http://europa.eu/rapid/press-release_IP-13-657_en.htm reached on Feb 19, 2018.

[17] When we talk about balancedness in this paper, we are not strictly talking about zero-balance conditions where imports and exports even each other out. The idea can also be relaxed in static and dynamic manners to attain an approximate balance over time. Indeed, there could be gains for intertemporal trades and our proposals also address these issues in Section 4.2.

rent form.[18] The Jesuit universities exchange program FACHEX is another one that is adversely affected. The program still does not have an explicitly embedded balancedness requirement. It includes all Jesuit universities but Georgetown, which is arguably the most prominent one.

The Erasmus student exchange program among universities in Europe is another example of a market in which a lack of balancedness have caused some exchange relationships to be terminated. Member colleges that want to exchange students with each other sign bilateral contracts that set the maximum number of students to be exchanged in certain years. The renewal decision of the contract depends on whether a reasonable balance is maintained between the incoming and outgoing exchange students between these colleges. In particular, if one of the colleges has more incoming students than its outgoing students, then that college does not renew the contract.

Tuition- and worker-exchange markets are closely related to favor markets, also known as "time banks," where time spent doing a favor or the number of favors is used as the currency of exchange. Holding of the transaction currency in such markets corresponds to a positive imbalance in our model. If not enough currency is injected initially to the system and there is too much uncertainty, agents may shy away from using their currency. Baby sitting co-ops are an example of such time banks. Such banks could be adversely affected by the lack of balanced clearing mechanisms that clear all favors through a well-defined schedule-matching scheme.[19]

3 Two-Sided Matching via Exchange: Model

Let C and S be the finite sets of **colleges** and **students**, respectively.[20] Set S is partitioned into $|C|$ disjoint sets, i.e., $S = \underset{c\in C}{\cup} S_c$ where S_c is the set of students who are applying to be sponsored by $c \in C$. Let $q = (q_c)_{c\in C} \in \mathbb{N}^{|C|}$ be the **(scholarship) admission quota** vector, where q_c is the maximum number of students who will be admitted by c with tuition exchange scholarship, and $e = (e_c)_{c\in C} \in \mathbb{N}^{|C|}$ is the **(scholarship) eligibility quota** vector, where $e_c \leq |S_c|$ is the number of students in S_c certified as eligible

[18] See https://www.insidehighered.com/news/2012/02/15/tuition-exchange-program-northwest-colleges-coming-end reached on Feb 17, 2018.

[19] In the mid-1970s, at the Capitol Hill Baby-Sitting Coop in Washington, DC, negative-balance aversion of families resulted in imbalances between families and decreased the number of favor exchanges between families. For details see http://www.ft.com/cms/s/2/f74da156-ba70-11e1-aa8d-00144feabdc0.html reached on Feb 20, 2016. This fits our setting perfectly: if the matches could be done in a monthly schedule using a centralized method, then balancedness requirements could be easily addressed.

[20] We will keep tuition exchange in mind in naming our concepts. The minor differences in the temporary worker-exchange model will be highlighted in Section 5.

students by c. Let $\rhd_C = (\rhd_c)_{c\in C}$ be the list of college **internal priority orders,** where $\rhd_c$ is a linear order over S_c based on some exogenous rule. We define the set of **eligible students** of c as $E_c = \{s \in S_c \mid r_c(s) \leq e_c\}$ where $r_c(s)$ is the rank of $s \in S_c$ under $\rhd_c$. Let $E = \underset{c\in C}{\cup} E_c$. Being unassigned option, named the *null college*, is denoted by $c_\emptyset$ and its quota is set as $q_{c_\emptyset} = |S|$.

To define the preferences properly, we define an auxiliary concept first: An **unconstrained matching** is a correspondence $\lambda : C \cup S \twoheadrightarrow C \cup S \cup c_\emptyset$ such that:[21] (i) $\lambda(c) \subseteq S$ for all $c \in C$, (ii) $\lambda(s) \subseteq C \cup c_\emptyset$ where $|\lambda(s)| = 1$ for all $s \in S$, and (iii) $s \in \lambda(c)$ if and only if $\lambda(s) = c$ for all $c \in C$ and $s \in S$. An outcome is a **matching**, which is an unconstrained matching μ satisfying $|\mu(c)| \leq q_c$ for all $c \in C$, and $\mu(s) = c_\emptyset$ for all $s \notin E$.[22]

Let $\mathcal{M}^u$ and $\mathcal{M}$ be the sets of unconstrained matchings and matchings, respectively. Given a fixed set of colleges C and students S, the set of unconstrained matchings is fixed across different admission and eligibility quotas, while the sets of matchings will change. Let $X_c^\mu = \{s \in S_c \mid \mu(s) \in C \setminus c\}$ be the **set of exports** for c in $\mu \in \mathcal{M}^u$.[23] Let $M_c^\mu = \{s \in S \setminus S_c \mid \mu(s) = c\}$ be the **set of imports** for c in $\mu \in \mathcal{M}^u$. We refer to $b_c^\mu \in \mathbb{Z}$ as the **net balance** of $c \in C$ in μ and define it as $b_c^\mu = |M_c^\mu| - |X_c^\mu|$. We say $c \in C$ has a *zero (negative)* [*positive*] net balance in μ if $b_c^\mu = 0$ ($b_c^\mu < 0$) [$b_c^\mu > 0$].

Let $\succsim = (\succsim_S, \succsim_C) = ((\succsim_s)_{s\in S}, (\succsim_c)_{c\in C})$ be the list of student and college **preferences over unconstrained matchings,** where $\succsim_i$ is the preference relation of agent $i \in S \cup C$. We denote the strict preference of $i \in S \cup C$ by $\succ_i$ and her indifference relation by $\sim_i$.

Each $s \in S$ cares only about her own match in an unconstrained matching and has a strict preference relation P_s on $C \cup c_\emptyset$. Let R_s denote the at-least-as-good-as relation associated with P_s for any student $s \in S$: cR_sc' if cP_sc' or $c = c'$ for all $c, c' \in C \cup c_\emptyset$. Student s's preferences over unconstrained matchings $\succsim_s$ is defined as follows: if $\mu(s)R_s\mu'(s)$ then $\mu \succsim_s \mu'$.

Each college potentially cares not only about its admitted class of (scholarship) students but also about its net balance. Each preference relation for a college is related to some strict ranking over sets of admitted students, i.e., subsets of S. Given a college c and preference $\succsim_c$ suppose P_c^* is this ranking. In turn, P_c^* is responsively induced through a linear order P_c over $S \cup \emptyset$. P_c^* is **responsive** to P_c if for all $T \subseteq S$ and $s, s' \in S \setminus T$: (1) $sP_c\emptyset \iff (T \cup s)P_c^*T$ and (2) $sP_cs' \implies (T \cup s)P_c^*(T \cup s')$.[24] Note that P_c^* is not

[21] We may refer to singleton $\{x\}$ as x with a slight abuse of notation. The only exception is $\{\emptyset\}$.

[22] In tuition exchange, only the students who are certified eligible can be assigned to other institutions. Therefore, if s is not certified eligible, i.e., if $s \in S \setminus E$, then she will be assigned to the null college.

[23] When we say $s \in S$ is matched to $c \in C$, we mean s receives a tuition-exchange scholarship from c.

[24] In the literature, property (1) is originally referred to as separability and (2) is the original respon-

the preference relation of c, but is a ranking over sets of admitted students. Let R^*_c be the weak ranking over the subsets of students induced by P^*_c. Throughout the paper, the relationship between preferences of c and this ranking is assumed as follows: between any two unconstrained matchings in which c has the same net balance, it prefers the one with the higher-ranked set of admitted students according, i.e., for any $\mu, \nu \in \mathcal{M}^u$, if $b^\mu_c = b^\nu_c$ and $\mu(c)\ R^*_c\ \nu(c)$ then $\mu \succsim_c \nu$. The domain of preferences for c include all such possible preferences $\succsim_c$.

Throughout the paper, C, S, and $\triangleright_C$ are fixed; a quota vector, an eligibility vector, and a preference profile defines a **tuition-exchange market** – or simply, a **market** – as $\left[q, e, \succsim\right]$.

We now introduce the properties of desirable matchings in a given market. A matching μ **Pareto dominates** $\nu \in \mathcal{M}$ if $\mu \succsim_i \nu$ for all $i \in C \cup S$ and $\mu \succ_j \nu$ for some $j \in C \cup S$. A matching μ is **Pareto efficient** if it is not Pareto dominated by any other $\nu \in \mathcal{M}$. A student s is **acceptable** for a college c if $s\ P_c\ \emptyset$, and c is **acceptable** for s if $c\ P_s\ c_\emptyset$. A matching μ is **acceptable** if it matches every agent with only acceptable partners. A matching μ is **balanced** if $b^\mu_c = 0$ for all $c \in C$.[25] Balancedness is the key property in tuition exchange. We say a balanced matching μ is **balanced-efficient** if it is not Pareto dominated by any other balanced matching.

We say $\mu \in \mathcal{M}$ is **blocked by a college** c if there exists some $\mu' \in \mathcal{M}$ such that $\mu' \succ_c \mu$, $\mu'(s) = \mu(s)$ for all $s \in S \setminus \mu(c)$, and $\mu'(c) \subset \mu(c)$. A matching μ is **blocked by a student** s if $c_\emptyset\ P_s\ \mu(c)$. A matching μ is **individually rational** if it is not blocked by any individual agent. A matching μ is **nonwasteful** if there does not exist a college-student pair (c, s) such that $|\mu(c)| < q_c$, $c\ P_s\ \mu(s)$ and $\mu' \succ_c \mu$ for some matching μ' where $\mu'(s) = c$ and $\mu'(s') = \mu(s')$ for all $s' \in S \setminus s$. In Online Appendix A, we provide an analysis of the decentralized practice of tuition exchange and "stability," defined there for a market with externalities.

3.1 Tuition-Exchange Mechanisms

The current practice of tuition exchange is implemented through indirect semi-decentralized market mechanisms. Although our new proposal can also be implemented indirectly, it will be useful to discuss it as a direct mechanism to analyze its properties. A **(direct) mechanism** is a systematic way of selecting a matching for each market. Let φ be a mechanism; then the matching selected by φ in market $\left[q, e, \succsim\right]$ is denoted by $\varphi\left[q, e, \succsim\right]$, and the assignment of agent $i \in S \cup C$ is denoted by $\varphi\left[q, e, \succsim\right](i)$.

siveness condition due to Roth (1985). We refer to the collection of both as responsiveness.

[25]Note that $b^\mu_c \geq 0$ for all $c \in C$ or $b^\mu_c \leq 0$ for all $c \in C$ each implies $b^\mu_c = 0$ for all $c \in C$.

In a revelation game, students and colleges report their preferences; additionally, colleges report their admission and eligibility quotas.[26] A mechanism φ is **immune to preference manipulation for students** (or **colleges**) if for all $\left[q, e, \succsim\right]$, there exists no $i \in S$ (or $i \in C$) and $\succsim'_i$ such that $\varphi\left[q, e, (\succsim'_i, \succsim_{-i})\right](i) \succ_i \varphi\left[q, e, \succsim\right](i)$. A mechanism φ is **immune to preference manipulation** if it is immune to preference manipulation for both students and colleges. A mechanism φ is **immune to quota manipulation** if for all $\left[q, e, \succsim\right]$, there exists no $c \in C$ and (q'_c, e'_c) with $q'_c \leq q_c$ such that $\varphi\left[(q'_c, q_{-c}), (e'_c, e_{-c}), \succsim\right](c) \succ_c \varphi\left[q, e, \succsim\right](c)$. A mechanism φ is **strategy-proof for colleges** if for all $\left[q, e, \succsim\right]$, there exists no $c \in C$ and $(q'_c, e'_c, \succsim'_c)$ with $q'_c \leq q_c$ such that $\varphi\left[(q'_c, q_{-c}), (e'_c, e_{-c}), (\succsim'_c, \succsim_{-c})\right](c) \succ_c \varphi\left[q, e, \succsim\right](c)$. A mechanism is **strategy-proof for students** if it is immune to preference manipulation for students. A mechanism is **strategy-proof** if it is strategy-proof for both colleges and students.[27] A mechanism φ is **group strategy-proof for students** if for all $\left[q, e, \succsim\right]$, there exists no $S' \subseteq S$ and $\succsim'_{S'} = (\succsim'_s)_{s \in S'}$ such that $\varphi\left[q, e, (\succsim'_{S'}, \succsim_{-S'})\right](s) \succsim_s \varphi\left[q, e, \succsim\right](s)$ for all $s \in S'$ and $\varphi\left[q, e, (\succsim'_{S'}, \succsim_{-S'})\right](s') \succ_{s'} \varphi\left[q, e, \succsim\right](s')$ for some $s' \in S'$.

One distinctive feature of tuition exchange is the existence of internal priorities for each $c \in C$, $\rhd_c$. In current practice, the internal priority order is used to determine which students will be certified eligible. This priority order is usually based on the seniority of faculty members. We incorporate this priority-based fairness objective into our model by introducing a new property. Formally, a mechanism φ **respects internal priorities** if whenever a student $s \in S_c$ is assigned to a college in market $\left[q, e, \succsim\right]$, then s is assigned to a weakly better college in $\left[q, (\tilde{e}_c, e_{-c}), \succsim\right]$ where $\tilde{e}_c > e_c$.[28] Respect for internal priorities is a fairness notion rather than efficiency.

4 Two-Sided Top Trading Cycles

In this section, we propose a mechanism that is individually rational, acceptable, balanced-efficient, and strategy-proof for students. Moreover, it respects colleges' internal priorities.

[26]Since the internal priority order is exogenous, the set of eligible students can be determined by the eligibility quota.

[27]Since students care only about the colleges they are matched with, it will be sufficient for them to report their preferences over colleges. Under an additional assumption, our proposal in Section 4 can also be implemented by having colleges only report individual students as "acceptable" or "unacceptable".

[28]This property is used in our characterization in Section 4 where we show that this axiom does not bring additional cost to our proposed mechanism (Theorem 6). Moreover, this property can be weakened as follows at no cost: a sponsored student who is matched to a college better than her outside option continues to be matched with a (possibly different) college better than her outside option when her home college increases the number of its sponsored students. The outside option for tuition exchange and worker exchange is null college and home firm, respectively.

Throughout our analysis, we impose a weak restriction on college preferences. Assumption 1 below states that a college prefers a better scholarship class with zero net balance to an inferior scholarship class with a nonpositive net balance.

Assumption 1 *For any $\mu, \nu \in \mathcal{M}^u$ and $c \in C$, if $b_c^\mu = 0$, $b_c^\nu \leq 0$, and $\mu(c) P_c^* \nu(c)$ then $\mu \succ_c \nu$.*

We start with the following proposition, which shows the incompatibility between balancedness and individual rationality, and nonwastefulness.

Proposition 1 *Under Assumption 1, there may not exist an individually rational and nonwasteful matching that is also balanced.*[29]

It will be useful to denote a matching as a directed graph, as we will find the outcome of our mechanism through an algorithm over directed graphs. In such graphs, colleges and students are nodes; a directed edge is between a college and a student, and it *points to* either the college or the student, but not both. Given a matching μ, let each $s \in S$ point to $\mu(s)$ and each $c \in C$ point to all its matched students, i.e., those in $S_c \setminus \mu(c_\emptyset)$; moreover, let $c_\emptyset$ point to students assigned to it. In this graph, we define the following subgraph: A **trading cycle** consists of an ordered list of agents $(c_1, s_1, c_2, s_2, ..., c_k, s_k)$ such that c_1 points to s_1, s_1 points to c_2,..., c_k points to s_k and, s_k points to c_1.

In the following Remark, we state that if a matching is balanced, then we can decompose it into a finite number of trading cycles. We skip its proof for brevity.

Remark 1 *A matching μ is balanced if and only if each student is in a trading cycle in the graph of the matching.*

We are ready to propose a new two-sided matching mechanism. We will find its outcome using an algorithm inspired by top-trading-cycles (TTC) introduced for one-sided resource allocation problems, such as for school choice (by Abdulkadiroğlu and Sönmez, 2003) and dormitory room allocation (by Abdulkadiroğlu and Sönmez, 1999). These TTC algorithms were inspired by *Gale's TTC algorithm* (Shapley and Scarf, 1974), which was used to find the core allocation of a simple exchange economy, referred to as the *housing market*, a subclass of one-sided matching problems. Most common mechanisms in one-sided matching problems function through algorithms that mimic agents exchanging objects that are initially allocated to them either through individual property rights or through the mechanism's definition of the agents (see also Pápai, 2000; Pycia and Ünver,

[29] All proofs are in Online Appendix B.

2017). In contrast, in our market, college slots are not objects. Therefore our definition of a mechanism, and the properties of matchings and mechanisms (except strategy-proofness for students) do not have any analogous translation in such problems. However, because we use a variant of TTC algorithm to find the outcome, we refer to our mechanism as ***two-sided (student-pointing) top-trading-cycles (2S-TTC)***. Its outcome is found for any given $\left[q, e, \succsim\right]$ as follows:[30]

The Algorithm for the Two-Sided Top-Trading-Cycles Mechanism:

Round 0: Assign two counters, for admission and eligibility, for each college $c \in C$, and set them equal to q_c and e_c, respectively.

Round $k \geq 1$***:*** Each available student points to her favorite among available colleges, which consider her acceptable, and $c_{\emptyset}$. Each available college c points to the highest priority available student in S_c according to $\triangleright_c$. Null college $c_{\emptyset}$ points to all students pointing to it. Due to the finiteness of C and S, there exists at least one cycle. Each agent can be part of at most one cycle. Every student in each cycle is assigned a seat at the option she is pointing to and removed. If the cycle does not contain $c_{\emptyset}$, then the counters of each college in that cycle are reduced by one. If the cycle contains $c_{\emptyset}$ and an eligible student from an available college $c \in C$, then we reduce only the eligibility counter of c by one. If any counter of a college reaches zero, then that college is removed.

The algorithm terminates when all students are removed.

In Theorem 1, we show that 2S-TTC is balanced-efficient, acceptable, and individually rational, and it respects internal priorities.

Theorem 1 *Under Assumption 1, 2S-TTC is an individually rational, balanced-efficient, and acceptable mechanism that also respects internal priorities.*

It should be noted that balanced-efficiency of 2S-TTC is not directly implied by (Pareto) efficiency of TTC in a one-sided market. Here, colleges are players with multiple seats. Observe that by assigning a college at least one highly preferred student and some unacceptable ones, some acceptable, individually rational, and balanced matchings can potentially be (weakly) improved for everyone while keeping balancedness intact (and even the number of students who are assigned to a college can go up). In this theorem, through an iterative approach, we show that it is not possible to improve over 2S-TTC's

[30] The converse of this process, using an algorithm originally introduced for two-sided matching markets in one-sided matching markets, has already been utilized in market design. For certain real-life one-sided problems regarding student placement and school choice, Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) introduced the student-optimal stable mechanism, whose algorithm was originally introduced to find stable matchings in two-sided matching markets by Gale and Shapley (1962). Later on, many school districts in the US adopted this mechanism for public school admissions (see Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005 and Abdulkadiroğlu, Pathak, and Roth, 2005).

outcome in such a fashion. Also consider the following concept: For any $I \subseteq S \cup C$, a balanced matching is **balanced-efficient for** I if there is no other balanced matching which makes each agent in I weakly better off and at least one agent in I strictly better off. 2S-TTC mechanism is neither balanced-efficient for students nor balanced-efficient for colleges. However, it is balanced-efficient overall (when all agents' welfare is taken into account). Thus, it is a compromise between the welfare of both sides, slightly favoring students by construction. Since side-balanced-efficiency is not satisfied by 2S-TTC in general (even under strict preferences), we need a new proof to prove its overall balanced-efficiency. To illustrate that 2S-TTC is not balanced-efficient for any side, we provide a simple example (Example 2) in Online Appendix G and further explanation regarding why previously known results do not immediately imply our efficiency result.

Under a centralized mechanism, incentives for participants to truthfully reveal their preferences are desirable. Unfortunately, we show that balanced-efficiency, individual rationality, and immunity to preference manipulation for colleges are incompatible.

Theorem 2 *There does not exist an individually rational (or acceptable) and balanced-efficient mechanism that is also immune to preference manipulation for colleges, even under Assumption 1.*

We prove this theorem by constructing several markets and showing that it is not possible to satisfy all three properties in one of these markets.

Theorems 1 and 2 imply that the 2S-TTC mechanism is not strategy-proof for colleges.

The following theorem shows that it is group strategy-proof for students. This result is a consequence of TTC being group strategy-proof in a housing market (see Pápai, 2000).

Theorem 3 *2S-TTC is group strategy-proof for students.*

2S-TTC can be run as an indirect mechanism where colleges report only their acceptable incoming students. Hence, the strategy space for the colleges is very simple in using 2S-TTC in the field: their strategy is to report their admission and eligibility quotas and their set of acceptable students based on their preferences, set of own students, and internal priority order.

Moreover, if we focus on the game played by the tuition-exchange offices of colleges, when admissions preferences are fixed, truthful admission quota revelation and certification of all their own students induces a dominant-strategy equilibrium under 2S-TTC.[31]

[31]On their websites, colleges explain that the sole reason for certifying a limited number of students is maintaining a balanced exchange. 2S-TTC removes the need for this rightful caution associated with the current market practices (see Online Appendix A).

Theorem 4 *Under Assumption 1 and when true eligibility quotas satisfy $e_c = |S_c|$ for all $c \in C$, 2S-TTC is immune to quota manipulation.*

We prove the theorem with a lemma showing that as the quotas of a college increase, the import and export sets and the admitted class of students of this college also (weakly) expand under 2S-TTC.[32]

Theorems 3 and 4 point out that only colleges can benefit from manipulation, and they can manipulate by misreporting their preferences. Moreover, the only way to manipulate preferences is to report an acceptable student as unacceptable. Suppose we take all the admitted students in the regular admission procedure as acceptable for a tuition-exchange scholarship. Then, to manipulate 2S-TTC, a college needs to reject a student who satisfies the college admission requirements. Usually college admission decisions are made before the applicants are considered for scholarships.[33]

Proposition 2 below implies that colleges do not benefit from misreporting their ranking over incoming classes.

Proposition 2 *Under Assumption 1, colleges are indifferent among strategies that report preferences in which the same set of students is acceptable with the same quota report under the 2S-TTC mechanism.*

We have shown that 2S-TTC has appealing properties. In the following theorem, we show that it is the unique mechanism satisfying a subset of these properties.

Theorem 5 *Under Assumption 1, 2S-TTC is the unique student-strategy-proof, acceptable, and balanced-efficient mechanism that also respects internal priorities.*

In the proof of our characterization theorem, we use a different technique from what is usually employed in elegant single quota characterization proofs such as Svensson (1999) and Sönmez (1995) for the result of Ma (1994). Our proof relies on building a contradiction with the claim that another mechanism with the four properties in the theorem's

[32]Theorem 4 is in stark contrast with similar results in the literature for stable mechanisms. The student- and college-optimal stable mechanisms are prone to admission quota manipulation by the colleges even under responsive preferences (see Sönmez, 1997 and Konishi and Ünver, 2006) Thus, 2S-TTC presents a robust remedy for a common problem seen in centralized admissions that use the student-optimal stable mechanism and also in tuition exchange in a decentralized market (see Theorems 10 and 11 in Online Appendix A).

[33]Our proposal also prevents some other manipulation possibilities. For example, right now if a college really likes one of its own students, then it may decrease its export quota preventing this student to be eligible, and the student, in the end, attends to her home college through tuition remission. However, in our proposal a college's export quota also determines the set of its own students who are eligible for tuition remission. Thus, no ineligible student can attend her home college through tuition remission. We think that tuition exchange and tuition remission programs should be run together (see Section 4.1).

hypothesis can exist. Suppose such a mechanism exists and finds a different matching than 2S-TTC for some market. The 2S-TTC algorithm runs in rounds in which trading cycles are constructed and removed. Suppose $S(k)$ is the set of students removed in Round k, while running 2S-TTC in such a way that in each round only one arbitrarily chosen cycle is removed and all other cycles are kept intact. We find a Round k and construct an auxiliary market with the following three properties: (1) Eligibility quotas of home colleges of students in $S(k)$ are set such that these are the last certified students in their respective home institutions; (2) all preferences are kept intact except those of students in $S(k)$, whose preferences are truncated after their 2S-TTC assignments; and (3) all students in $S(k)$ are assigned $c_\emptyset$ under the alternative mechanism, while all students removed in the 2S-TTC algorithm before Round k have the same assignment under 2S-TTC and the alternative mechanism. This contradicts the balanced-efficiency of the alternative mechanism: we could give the students in $S(k)$ their 2S-TTC assignments while keeping all other assignments intact and obtain a Pareto-dominating balanced matching. Round k and the auxiliary market are constructed in three iterative steps.

Among all the axioms, only the respect for internal priorities is based on exogenous rules. One might suspect that more students will benefit from the tuition-exchange program if we allow the violation of respect for internal priorities. However, such mechanisms turn out to be manipulable by students.

Theorem 6 *Any balanced and individually rational mechanism that does not assign fewer students than 2S-TTC and selects a matching in which more students are assigned whenever such a balanced and individually rational outcome exists, is not strategy-proof for students, even under Assumption 1.*

4.1 Market Implementation: Tuition Remission and Exchange

Incorporating tuition-remission programs by all participating colleges in tuition exchange is the best way to implement a centralized clearinghouse. If parallel remission and exchange programs are run, as in current practice, a student may receive multiple scholarship offers, one from her home college and one from the tuition-exchange program. If the student accepts the home college's offer, the net balance of the college may deteriorate.

Although the current system is inflexible in accommodating this important detail, a clearinghouse utilizing 2S-TTC can easily combine tuition exchange with remission. Indeed, in Assumption 1, we allowed a college to deem its own sponsored students to be acceptable. Hence, all our results in this section robust to integration.

More specifically, we propose to run an indirect version of 2S-TTC in sequential stages in a semi-decentralized fashion: first, colleges announce their tuition-exchange scholarship quotas and which of their students are eligible to be sponsored for both exchange and remission; then, eligible students apply for scholarship to the colleges they find acceptable; then colleges send out scholarship admission letters. At this stage, as students have also learned their opportunities in the parallel-running regular college admissions market, they can form better opinions about the relative ranking of the null college, i.e., their options outside the tuition-exchange market. Students submit rankings over the colleges that admitted them with a tuition-exchange scholarship and the relative ranking of their outside option. Finally, 2S-TTC is run centrally to determine the final allocation.

4.2 Allowing Tolerable Imbalances

Some programs care about approximate balance over a moving time window. Here, we relax the zero-balance constraint and allow each $c \in C$ to maintain a balance within an interval $[\ell_c, u_c]$ where $\ell_c \leq 0 \leq u_c$.[34] When either ℓ_c or u_c equals zero for all $c \in C$, the market turns into the case studied in Section 4. Let $(\ell_c, u_c)_{c \in C}$ be the tolerance profile.

When the colleges hold a non-zero balance, then there may exist some colleges exporting (importing) more than they import (export). Then, we cannot represent all allocations by cycles. Therefore we need to consider chains in addition to the cycles. A **chain** is an ordered list $(c_1, s_1, c_2, s_2, ..., c_k)$ such that c_1 points to s_1, s_1 points to c_2,..., c_{k-1} points to s_{k-1} and s_{k-1} points to c_k. We refer to c_1 as the tail and c_k as the head of the chain.

We use a mechanism similar to 2S-TTC referred as the *two-sided tolerable top-trading-cycles*. For any market and tolerance profile, its outcome is found as follows:

Two-Sided Tolerable Top Trading Cycles (2S-TTTC):

Step 0: Fix an exogenous priority order among colleges. Assign two counters for each $c \in C$, o_c^q and o_c^e, and set them equal to q_c and e_c, respectively. Let b_c track the current net balance of c in the fixed portion of the matching. Initially set $b_c = 0$ for each $c \in C$. All colleges are marked as **importing** and **exporting**.

Step 1a:

* If $o_c^e = 0$, and either $o_c^q = 0$ or $b_c = u_c$, then remove c. If $o_c^e = 0$, $o_c^q > 0$, and $b_c < u_c$, then c becomes **non-exporting.**[35]

* If $o_c^q = 0$ and $b_c = \ell_c$, then remove c. If $o_c^q = 0$, $o_c^e > 0$, and $b_c > \ell_c$ then c becomes **non-importing**.

Step 1b: Each available student points to her favorite among *available importing* colleges,

[34]Here, ℓ_c and u_c are integers.

[35]I.e., a college is non-exporting if it has available quota to import but all its sponsored students are removed. Therefore, a non-exporting college cannot point to a student.

which consider her acceptable, and $c_\emptyset$. Each *available exporting* college c points to the highest priority available student in S_c according to $\triangleright_c$. Null college $c_\emptyset$ points to all students pointing to it.

Proceed to Step 2 if there is no cycle. Otherwise, in each cycle assign each student to the option she is pointing to and remove her. For each cycle and college c:

* Reduce eligibility counter o^e_c by one if it has an eligible student in that cycle.

* Reduce import counter o^q_c by one if it is in that cycle.

* Return to Step 1a.

Step 2: If there are no *exporting* colleges left,[36] then the algorithm terminates. If not, then we consider chains that end with *non-exporting* colleges.[37] If $b_c = \ell_c$ for each *available exporting* college c, then remove all *non-exporting* colleges and go to Step 1a.[38] Otherwise, find among the considered chains the one whose tail has the highest priority among the *available exporting* colleges c with $b_c > \ell_c$. Assign each student in that chain to the college that she points to and remove her. Denote the tail and head of the chain by c_t and c_h, respectively. Observe that c_h is a non-exporting college. Other colleges in the chain are represented by $\tilde{c}$:

* Reduce eligibility counter $o^e_{\tilde{c}}$ and import counter $o^q_{\tilde{c}}$ of all $\tilde{c}$ by one.

* Reduce eligibility counter $o^e_{c_t}$ and current net balance b_{c_t} by one.

* Reduce import counter $o^q_{c_h}$ by one and increase current net balance b_{c_h} by one.

* Return to Step 1a.

When the algorithm terminates, all remaining students are assigned to $c_\emptyset$. We call each repetition of these two steps a round.

The 2S-TTTC mechanism inherits the most desired features of 2S-TTC. We state two theorems to this end.

Theorem 7 *2S-TTTC is strategy-proof for students and for any market $[q, e, \succsim]$ and tolerance profile $(\ell_c, u_c)_{c\in C}$, there does not exist an acceptable matching ν that Pareto dominates the outcome of 2S-TTTC and $\ell_c \leq b^\nu_c \leq u_c$ for all $c \in C$.*

In the 2S-TTTC mechanism, a student is pointed to by the colleges in C after all the other students with higher internal priority are assigned to a college or $c_\emptyset$. Moreover, a student points only to the acceptable colleges that also consider her acceptable. As a consequence, the 2S-TTTC mechanism satisfies acceptability and respect for internal priorities.

Theorem 8 *2S-TTTC is acceptable and it respects internal priorities.*

[36]Note that this condition also captures the case "if no eligible students are left".

[37]If no student points to an available non-exporting college, then we would have a cycle.

[38]That is, no more chains respecting the tolerance profile can form after this point in the algorithm.

Theorems 7 and 8 hold without any assumptions on preferences. Under a mild assumption on college preferences, we can show that 2S-TTTC is individually rational and it induces a dominant-strategy equilibrium for colleges' quota reporting game to certify all their students and report their true admission quota.

Although 2S-TTTC is defined in a static problem, we can easily extend it to the dynamic environment where the aggregate balance over years matters. In particular, for each period t and $c \in C$ we can set counter b_c equal to c's aggregate balance in period $t-1$ where the aggregate balance in period $t-1$, is equal to the sum of balances between period 1 and $t-1$. Moreover, the exogenous priority rule used in period t can be determined based on the aggregate balance colleges carry at the end of period $t-1$ such that the highest priority can be given to the college with the highest aggregate balance and so on.

5 Temporary Worker Exchanges

Many organizations have temporary worker-exchange programs that can be modeled through our balanced two-sided matching framework. The first difference between such programs and tuition exchange is that these exchanges are usually temporary. Each firm usually requires a set of specific skills, e.g., a mathematics teacher to replace their own mathematics teacher. Compatibility and ability to perform the task are the main preference criterion rather than a strict preference ranking. E.g., finding a good teacher with a specific degree is the first-order requirement, rather than finer details about the rankings of all good teachers.

The second difference is that each position and each worker should be matched, unlike the tuition-exchange application. The workers are currently working for their home firms. Thus, the firms consider these workers necessarily acceptable. By contrast, in tuition exchange, colleges are not required to admit all the dependents of their employees. In temporary worker exchanges, a worker who does not want to go to a different firm necessarily stays employed in her home firm. We need to use a variant of the tuition-exchange model to facilitate balanced-efficient trade in such circumstances.

We can use the model introduced in Section 3 with slight changes. Since each firm accommodates its current workers, $q_c = |S_c|$ for each $c \in C$. In Section 3, in the definition of a matching, students who are not eligible are taken as assigned to $c_\emptyset$. However, for worker-exchange programs, the workers who are not certified as eligible continue to work in their home firms in a matching . Formally, a **matching** is a correspondence $\mu : C \cup S \twoheadrightarrow C \cup S$ such that, (1) $\mu(c) \subseteq S$, where $|\mu(c)| = q_c = |S_c|$ for all $c \in C$, (2) $\mu(s) \subseteq C$, where $|\mu(s)| = 1$ for all $s \in S$, (3) $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in C$ and $s \in S$, and

(4) $\mu(s) = c$ for all $s \in S_c \setminus E$ and $c \in C$. Observe that each matching is balanced in this environment by definition. Thus, balanced-efficiency and Pareto efficiency are equivalent.

To capture the features of worker-exchange programs, we make certain assumptions about the preferences of workers and firms. Since worker $s \in S_c$ is already working at firm c, we assume that s finds c acceptable and c finds s acceptable, i.e., $cP_s c_\emptyset$ and $sP_c\emptyset$ for all $s \in S_c$ and $c \in C$. As discussed above, acceptable workers do not have huge differences for the firms. The compatibility assumption and Assumption 1 together imply that each firm weakly prefers an unconstrained matching with zero net balance to another unconstrained matching with nonpositive balance as long as it gets weakly more acceptable workers under the former one. We formally state these assumptions on preferences as follows.

Assumption 2 *(1) (Weakly size-monotonic firm preferences) For any $c \in C$ and $\mu, \nu \in \mathcal{M}^u$, if $b_c^\mu = 0$, $b_c^\nu \leq 0$, and $|\{s \in \mu(c) : sP_c\emptyset\}| \geq |\{s \in \nu(c) : sP_c\emptyset\}|$ then $\mu \succsim_c \nu$, and (2) (acceptability of current match) for any $c \in C$ and $s \in S_c$, we have $cP_s c_\emptyset$ and $sP_c\emptyset$.*

Based on Assumption 2, a (balanced) mechanism that allows employees to get better firms, which consider them acceptable, improves the total welfare without hurting anyone. Hence, 2S-TTC can be applied to temporary exchange programs with a minor change such that when a firm c is removed, all its remaining workers are assigned to it.[39] In this environment, 2S-TTC inherits its desired features. Moreover, the characterization result also holds. Additionally, acceptability of 2S-TTC implies that it is strategy-proof for firms.

Theorem 9 *Under Assumption 2, 2S-TTC is a (balanced) Pareto efficient, individually rational, acceptable, and strategy-proof mechanism that also respects internal priorities, and it is the unique Pareto efficient, acceptable, worker-strategy-proof mechanism that also respects internal priorities.*[40]

6 Conclusions

This paper proposes a centralized market solution to overcome problems observed in decentralized exchange markets. We used tuition exchange and temporary exchange programs as our leading examples, in which more than 300,000 people participate annually.

[39]Since $q_c = |S_c|$ for all $c \in C$, a firm is removed when its eligibility counter reaches to zero.

[40]Moreover, 2S-TTC is stable in this domain. This result is noteworthy, because the widely-used worker-proposing deferred-acceptance mechanism with exogenous tie-breaking favoring own workers over the others is not Pareto efficient, although it is stable and balanced in this special environment. If tie-breaking does not favor own workers the outcome of DA may not be a "matching" in this domain. In the proof of Theorem 9 we also show that 2S-TTC is stable in this domain.

Our paper, besides introducing a new applied problem and proposing a solution to it, has six main theoretical and conceptual contributions: We introduce a new two-sided matching model that builds on the two most commonly used matching models in the literature: discrete object allocation, including school choice, and standard many-to-one two-sided matching models, but differs in many fronts from these. As far as we know, this is the first time object allocation and exchange algorithms inspire the mechanism design for a two-sided matching model. This is one of the few instances when axiomatic mechanism design is used in practical market design to come up with the correct mechanism. A natural axiomatic representation is given for a TTC-based mechanism. This is one of the rare occasions where the stable matching theory of Gale and Shapley is extended to a setting with externalities with tractable existence, equilibrium, and comparative-static results (see Online Appendix A). Finally, our paper is one of the few studies that propose a dynamic matching mechanism with good properties for a dynamic applied problem.

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Supplementary Material for "Two-Sided Matching via Balanced Exchange" by Umut Mert Dur and M. Utku Ünver

Appendix A On Current Practice of Tuition Exchange

In this appendix, we analyze the current practice of tuition exchange. As the centralized process is loosely controlled, once each college sets its eligibility/admission quota and eligible students are determined, the market functions more like a decentralized one rather than centralized. Once colleges commit to the students they will sponsor, they lose their control over them. A sponsored student can sometimes get multiple offers and decide which one to accept and when to accept it. Hence, *stability* emerges as a relevant notion for a benchmark market-equilibrium concept when there is no other friction. To adopt stability in our model, we introduce blocking by a pair: Given a market $\left[q, e, \succsim\right]$, we say matching μ' is obtained from matching μ by the **mutual deviation** of c and s if $s \in \mu'(c) \subseteq \mu(c) \cup s$, and $\mu'(s') = \mu(s')$ for all $s' \in S \setminus (\mu(c) \cup s)$. A matching μ is **blocked by college-student pair** (c, s) if $c \; P_s \; \mu(s)$ and $\mu' \succ_c \mu$ for some matching μ' obtained from μ by the mutual deviation of c and s. As in any blocking condition in cooperative games with externalities, we need to take a stance on how other players act when a pair deviates. We assume that only a college, a student or a college-student pair deviates at a time, and assume that the rest of the students and colleges do not make simultaneous decisions.[1] A matching μ is **stable** if it is individually rational and not blocked by any college-student pair.

Tuition exchange has some idiosyncratic properties different from those of previously studied two-sided matching markets.

In tuition exchange, an admitted class of lower-quality students can be preferable to one with higher-quality students under two different matchings, if the latter one deteriorates the net balance of the college. The extreme version of this preference is a college being extremely averse against negative net-balance matchings, regardless of the incoming class, because maintaining a nonnegative net balance is important for a college to continue its membership in the program.

We will incorporate these features as two formal assumptions in this section. As-

[1]See Pycia and Yenmez (2015) for more discussion of this stability concept under matching problems with externalities.

sumption 3 states that a better admitted class is preferable as long as the net balance does not decrease, admission of unacceptable students deteriorates the rankings of unconstrained matchings regardless of their net balances, and a college deems its own students unacceptable in tuition exchange. Assumption 4 introduces negative net-balance averse preferences. In all results in this section we will use Assumption 3, while Assumption 4 will be used in only one result. We start by stating Assumption 3.

Assumption 3 *For any $c \in C$ and $\mu, \nu \in \mathcal{M}^u$,*

(1) (preference increases with a better admitted class and a non-deteriorating balance) if $b_c^\mu \geq b_c^\nu$ and $\mu(c) P_c^* \nu(c)$, then $\mu \succ_c \nu$,

(2) (awarding unacceptable students exchange scholarships is not preferable) if there exists $s \in \nu(c) \setminus \mu(c)$, $\emptyset P_c s$ and $\nu(s') = \mu(s')$ for all $s' \in S \setminus s$, then $\mu \succ_c \nu$, and

(3) (unacceptability of the college's own students for exchange scholarships) $\emptyset P_c s$ for all $s \in S_c$.

Assumption 3 implies that, if there exists $s \in \mu(c)$ such that $\emptyset P_c s$, then matching μ is blocked by c. Moreover, if $s P_c \emptyset$ for all $s \in \mu(c)$, then matching μ is not blocked by c. Hence, individual rationality and acceptability are equivalent under Assumption 3. Moreover, Assumption 3 implies that if $c P_s \mu(s)$, $s P_c \emptyset$, and $|\mu(c)| < q_c$, then (c, s) is a blocking pair for matching μ. Similarly, if $s P_c s'$, $s P_c \emptyset$, $s' \in \mu(c)$, and $c P_s \mu(s)$, then (c, s) is a blocking pair for matching μ.

The existence of stable matchings has been widely studied in two-sided matching problems without externalities. For instance, in the college admission market, when the college preferences are responsive up to quota, then the set of stable matching is nonempty (see Gale and Shapley, 1962; Roth, 1985).[2] We prove a similar result for our environment.

Proposition 3 *Under Assumption 3, there exists at least one stable matching in any tuition-exchange market.*

[2]In the earlier two-sided matching literature, stability a la Gale and Shapley (1962) has been the central solution concept. Technically, our model is similar to a two-sided matching model with externalities, i.e., agents have preferences over allocations rather than their matches. Sasaki and Toda (1996) introduce externalities in two-sided matching markets and various stability definitions. Pycia (2010) explores existence in two-sided matching when agents have preferences over peers and matches. The first model is quite general; however, their stability notion, which guarantees existence, requires a very conservative definition of blocking. The second model, on the other hand, does not cover externalities regarding the balancedness requirement. Pycia and Yenmez (2015) also focus on the existence of stable matching in a two-sided matching problem with externalities such that preferences satisfy a substitutes condition.

However, our model has major differences from standard externality models, which generally inspect peer effects or induce different stability definitions as a solution for the decentralized market. We use the standard stability notion in a model with externalities.

We prove this proposition by constructing an associated Gale-Shapley college-admissions market in which the set of Gale-Shapley-stable matchings is identical to the set of stable tuition-exchange matchings.

In Section 4, we showed the incompatibility between individual rationality, nonwastefulness, and balancedness under Assumption 1. Although Assumption 3 is stronger than Assumption 1, the incompatibility result still holds under Assumption 3.

Proposition 4 *Under Assumption 3, there may not exist an individually rational and nonwasteful matching that is also balanced.*

Proposition 4 also shows that there exists no stable and balanced mechanism under Assumption 3. One can then wonder whether there exists a stable mechanism that performs better than all other stable mechanisms in terms of balancedness. We prove otherwise.[3]

Proposition 5 *Under Assumption 3, each college has the same net balance in all stable matchings in a given market.*

We also investigate what kinds of strategic decisions a tuition-exchange office in a college would face in a quota-determination game if a stable outcome emerges in the market. Here we explicitly make the aforementioned additional assumption about negative net-balance aversion on college preferences:[4]

Assumption 4 *(Negative Net-Balance Aversion) Any college $c \in C$ prefers $\mu \in \mathcal{M}^u$, such that $b_c^\mu = 0$ and all $s \in \mu(c)$ are acceptable, to all $\nu \in \mathcal{M}^u$ with $b_c^\nu < 0$.*

In the quota-determination game, we fix C, S, $\rhd_C$, and $\succsim$. Colleges are the players of the game and each college's strategy is setting its admission and eligibility quotas under a simultaneous move, complete information setting. Without loss of generality, we constrain the strategy space such that a reported admission quota is not less than the reported eligibility quota. Given a true quota profile, denote the action set for c with A_c; then, it is $A_c = \left\{(\hat{q}_c, \hat{e}_c) \in \mathbb{N}^2 \mid \hat{q}_c \geq \hat{e}_c \geq 0\right\}$. The outcome of the game is determined by a stable mechanism (solution). In Theorem 10, by using the results of Proposition 6 below, we show that in any stable solution, if a college holds a negative net balance, then the best response is only to decrease its eligibility quota. Proposition 6 also gives us a

[3]We also inspect the structure of stable matchings in Appendix F. We show that there always exist college- and student-optimal stable matchings.

[4]This assumption is used only in Theorem 10.

comparative result regarding how the net balances of colleges change when they certify one additional student and do not decrease their admission quotas.[5]

Proposition 6 *Under Assumption 3, for fixed preferences* $\succsim$ *and for any reported quota profiles* $\hat{q}$ *and* $\hat{e}$*, let* $\hat{\pi}$ *and* $\tilde{\pi}$ *be stable matchings for the induced markets* $\left[\hat{q}, \hat{e}, \succsim\right]$ *and* $\left[(\tilde{q}_c, \hat{q}_{-c}), (\tilde{e}_c, \hat{e}_{-c}), \succsim\right]$*, respectively, where* $\hat{q}_c \geq \hat{e}_c$*,* $\tilde{q}_c \geq \hat{q}_c$ *and* $\tilde{e}_c = \hat{e}_c + 1$*. Then* $b_c^{\tilde{\pi}} \in \{b_c^{\hat{\pi}} - 1, b_c^{\hat{\pi}}\}$ *if* $b_c^{\hat{\pi}} < 0$*; and* $b_c^{\tilde{\pi}} \in \{b_c^{\hat{\pi}} - 1, b_c^{\hat{\pi}}, ..., b_c^{\hat{\pi}} + \tilde{q}_c - \hat{q}_c\}$ *if* $b_c^{\hat{\pi}} \geq 0$*.*

The proposition concludes that, when a college increases its eligibility quota by one without decreasing its admission quota, its overall net balance will decrease at most by one under any stable solution. Its net balance may increase only if it is a nonnegative net-balance college to start with.[6]

Theorem 10 *Under Assumptions 3 and 4, for fixed preferences* $\succsim$ *and for any reported quota profiles* $\hat{q}$ *and* $\hat{e}$*, if* c *has a negative net balance in a stable matching for market* $\left[\hat{q}, \hat{e}, \succsim\right]$ *where* $\hat{q}_c \geq \hat{e}_c$*, then its best response in any stable solution is to set only lower* $\hat{e}_c$*, but not higher; and in particular, there exist* $\tilde{e}_c \leq \hat{e}_c$ *such that college* c *has a zero-balance in every stable matching of the market* $\left[\hat{q}, (\tilde{e}_c, \hat{e}_{-c}), \succsim\right]$*.*

Theorem 10 shows that if c has a negative net balance then it certifies fewer students, which will eventually increase its balance.[7] When c certifies fewer students it may cause another college c' to have a negative net balance. Then c' will have a negative net balance and will certify fewer students, too. In Theorem 11 below, we show this result.

Theorem 11 *Under Assumption 3, for fixed preferences* $\succsim$ *and for any reported quota profiles* $\hat{q}$ *and* $\hat{e}$*, if a college* c *is holding a negative net balance in a stable matching* μ *for market* $\left[\hat{q}, \hat{e}, \succsim\right]$ *such that* $\hat{q}_c \geq \hat{e}_c$*, then* $b_{-c}^{\mu} \geq b_{-c}^{\mu'}$ *where* μ' *is any stable matching for market* $\left[(q'_c, \hat{q}_{-c}), (e'_c, \hat{e}_{-c}), \succsim\right]$*, and* $\hat{q}_c \geq q'_c \geq \hat{e}_c - 1 \geq e'_c$*.*

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[5]Weber (1997); Engelbrecht-Wiggans and Kahn (1998); Ausubel, Cramton, Pycia, Rostek, and Weretka (2014) study demand reduction in auctions.

[6]This is possible only if $\tilde{q}_c > \hat{q}_c$.

[7]This result is in a similar vein as the results on college admissions where the DA mechanism is shown to be prone to admission quota manipulation of the colleges under responsive preferences, regardless of imbalance aversion (see Sönmez, 1997). However, Konishi and Ünver (2006) show that the DA mechanism would be immune to quota manipulation, if preferences of colleges over incoming students were responsive and monotonic in number. On the other hand, even under this restriction of preferences over the incoming class, our result would imply all stable mechanisms are manipulable with quota reports for colleges with negative net balances if colleges have negative net-balance averse preferences. (See also Kojima and Pathak, 2009.)

Theorems 10 and 11 do not conduct an equilibrium analysis in a quota-determination game. But they do point out that in a frictionless market, the colleges that will be likely to have a negative-balance will be conservative and will decrease their eligibility quotas for exports, which will further deteriorate the balances of other colleges.

Typically, no college fully withdraws in practice, as there is often a minimum quota of participation in place. We conjecture that this could be instituted because of the reasons outlined above. Given that continued membership is an attractive benefit, often times, smaller colleges will announce that they will import and export at this minimum quota requirement, and will continue to be a member of the program without fully withdrawing from the system.

We conclude that under a new design for tuition exchange, there should be no room for quota underreporting by the colleges due to negative net-balance aversion, if possible. A fully centralized solution disregarding decentralized market stability seems to be inevitable, as stability is at odds with balancedness and has various other shortcomings regarding other incentives.

Moreover, we deem such a stability concept inappropriate for our purpose as the rights of students to participate in market activity depends on the permission of their colleges. Thus, we claim that balanced-efficiency and individual rationality are the most important features of a tuition-exchange outcome.

Appendix B Proofs

Proof of Propositions 1 and 4. Consider the following market. Let $C = \{a, b\}$ and for each $c \in C$ set $q_c = e_c = 1$. The set of students in each college is: $S_a = \{\mathbf{1}\}$ and $S_b = \{\mathbf{2}\}$. The associated strict preference relations of students over colleges are given as $P_{\mathbf{1}} : b \ P_{\mathbf{1}} \ c_\emptyset \ P_{\mathbf{1}} \ a$ and $P_{\mathbf{2}} : a \ P_{\mathbf{2}} \ c_\emptyset \ P_{\mathbf{2}} \ b$. College preferences satisfy Assumption 1 (Assumption 3). Student $\mathbf{1}$ is not acceptable to b, i.e., $\emptyset \ P_b \ \mathbf{1}$, and b prefers any matching in which no student is assigned to itself over the matchings in which $\mathbf{1}$ is assigned to itself. Student $\mathbf{2}$ is acceptable to a and a prefers any matching with positive balance to the matchings in which no student is assigned to itself. There is one nonwasteful matching that is not individually blocked: $\mu(\mathbf{1}) = c_\emptyset$ and $\mu(\mathbf{2}) = a$. This matching is not balanced, as college b has negative net balances in μ. ■

Proof of Theorem 1. Consider an arbitrary market $\left[q, e, \succsim\right]$. Let π be the matching selected by 2S-TTC for $\left[q, e, \succsim\right]$. Let E be the set of eligible students in $\left[q, e, \succsim\right]$. First note that, $\pi(s) = c_\emptyset$ for all $s \in S \setminus E$. In particular, under 2S-TTC $s \in S \setminus E$ is never pointed to by her home college. Hence, 2S-TTC selects a matching for $[q, e, \succsim]$.

Acceptability: Students will be assigned to null college $c_\emptyset$ whenever they point to it, and, hence, they will never need to point to an unacceptable college. Hence, a student cannot be assigned to an unacceptable college. Moreover, a student cannot point to a college that considers her unacceptable. Therefore, the students ranked below $\emptyset$ in P_c cannot be assigned to c. Thus, 2S-TTC is acceptable.

Individual Rationality: Since each $s \in S$ is assigned to an option (weakly) better than $c_\emptyset$, s does not individually block π. Since all students in $\pi(c)$ are ranked above $\emptyset$ in P_c for each $c \in C$, $\pi(c)\ R_c^*\ \tilde{S}$ for any $\tilde{S} \subseteq \pi(c)$. In any matching μ such that $\mu(s) = \pi(s)$ for all $s \in S \setminus \pi(c)$ and $\mu(c) \subset \pi(c)$, $c \in C$ has a nonpositive net balance. Hence, π is not individually blocked by c.

Respect for Internal Priorities: Suppose, contrary to the claim, that 2S-TTC does not respect internal priorities. Then, there exists $s \in S_c$, who is assigned to a college by 2S-TTC in $\left[q, e, \succsim\right]$, is assigned to a worse option in $\left[q, (\tilde{e}_c, e_{-c}), \succsim\right]$ where $\tilde{e}_c > e_c$. Since any ineligible student is assigned to $c_\emptyset$ in any market, $e_c > 0$, $q_c > 0$ and $r_c(s) \leq e_c$.[8] We use a variation of the 2S-TTC in which only the students with the highest internal priority at their home colleges point to a college each round. Since only the top-priority students and students pointing to $c_\emptyset$ can form a cycle in each round under both versions of 2S-TTC, they will select the same outcome. Let $S(k)$ and $\tilde{S}(k)$ be the set of students in the cycles removed in Round k of 2S-TTC applied to the markets $\left[q, e, \succsim\right]$ and $\left[q, (\tilde{e}_c, e_{-c}), \succsim\right]$, respectively.[9] In both markets, the same set of students will be active, i.e. point to a college in C or $c_\emptyset$, in the first round. Since we consider the same preference profile, $S(1) = \tilde{S}(1)$. Then, if $s \in S(1)$, she is assigned to the same college in both markets. If not, consider the second round. Since the same set of students is removed with their assignments and $s \notin S(1)$, the set of active students and the remaining colleges in the second round of 2S-TTC applied to the either market will be the same. Moreover, students will be pointing to the same options in both markets. Hence, $S(2) = \tilde{S}(2)$. Then, if $s \in S(2)$, she is assigned to the same colleges in both markets. If not, we can repeat the same steps and show that s will be assigned to her match in $[q, e, \succsim]$ in the outcome of 2S-TTC in market $\left[q, (\tilde{e}_c, e_{-c}), \succsim\right]$.

Balanced-efficiency: Since the matching selected by 2S-TTC consists of trading cycles in which students and their assignments form unique cycles, its outcome is balanced by Remark 1. Since 2S-TTC is acceptable, π is also acceptable. Let $S(k)$ be the set of

[8] For the proof of Theorem 9, note that in any market each worker is assigned to a firm weakly better than her home firm and all ineligible workers are assigned to their home firms. Hence, $e_c > 0$, $q_c > 0$ and $r_c(s) \leq e_c$.

[9] For the proof of Theorem 9, since s is an eligible worker in market $\left[q, e, \succsim\right]$, $s \in S(k')$ for some $k' \leq K$ where K is the last round of 2S-TTC in market $\left[q, e, \succsim\right]$.

students who are in the cycles removed in Round $k \le K$ of 2S-TTC where K is the last round of 2S-TTC.[10] We will prove that π is balanced-efficient in two parts.

Part I: We first prove that π cannot be Pareto dominated by another acceptable balanced matching. If $s \in S(1)$, then $\pi(s) \in C \cup c_\emptyset$ is the highest ranked option in P_s that considers her acceptable. That is, no student $s \in S(1)$ can be assigned to a better college considering her acceptable. If there exists a matching ν such that $\nu \succ_s \pi$, then $\nu(s)$ considers s unacceptable. That is, π cannot be Pareto dominated by another acceptable matching ν in which at least one student in $S(1)$ is better off in ν.

If a student $s \in S(2)$ is not assigned to a more preferred $c \in C$ that considers her acceptable, then c should be removed in Round 1. Let ν be an acceptable and balanced matching such that $\nu(s) = c$. Suppose there exists another student s' such that $\pi(s') = c$ and $\nu(s') \neq c$. Note that s' is an eligible student. Because s' is assigned in Round 1, $\pi(s') = c$ is her favorite college among the ones considering her acceptable. That is, in any acceptable and balanced matching ν in which s is assigned to $\pi(s')$, s' will be made worse off. Suppose $\nu(s') = c$ for any $s' \in \pi(c)$.[11] Then, c is removed in Round 1 since its eligibility counter reaches to zero and $s \notin S_c$. Balancedness of ν implies that there exists a student $\tilde{s} \in E_c \cap S(1)$ such that $\pi(\tilde{s}) = c_\emptyset$ and $\nu(\tilde{s}) \in C$. Then, ν cannot be acceptable, because $\tilde{s}$ considers all colleges considering her acceptable as unacceptable. Hence, π cannot be Pareto dominated by another balanced and acceptable matching ν in which at least one student in $S(2)$ is better off in ν. In particular, if a student in $S(2)$ prefers ν to π, then at least one student in $S(1)$ prefers π to ν.[12]

We similarly show the same for all other rounds of 2S-TTC. Thus, in a balanced matching no student can be assigned to a better college among the colleges that consider her acceptable without harming another student or violating balancedness or feasibility constraints. Hence, no college can be made better off without harming another agent either, if we focus on matchings that are acceptable and balanced.

Part II: Next we show that there does not exist an unacceptable balanced matching that Pareto dominates π. To the contrary of the claim, suppose there exists an unacceptable balanced matching ν that Pareto dominates π. By definition, $\pi(s) = \nu(s) = c_\emptyset$ for any $s \notin E$.[13] Then each $i \in C \cup S$ weakly prefers ν to π, and at least one agent $j \in C \cup S$ strictly prefers ν to π. Due to the acceptability of the 2S-TTC, every student weakly prefers her assignment in π to $c_\emptyset$. Therefore, every assigned student in π is also

[10]For the proof of Theorem 9, note that any worker removed after the removal of her home firm is ineligible and she is assigned to her home firm in any matching.

[11]For the proof of Theorem 9, this case is not possible because in 2S-TTC's matching each firm fills its all seats.

[12]We use this fact also in the proof of Theorem 5.

[13]For the proof of Theorem 9, $\pi(s) = \nu(s) = c$ for all $s \in S_c \setminus E$ and $c \in C$.

assigned to an acceptable college in ν. Thus, due to the balancedness of both π and ν, $|\nu(c)| \geq |\pi(c)|$ for all $c \in C$.[14] As ν is unacceptable, there exists some $c_0 \in C$ such that $s_0 \in \nu(c_0)$ is unacceptable for c_0.[15] As $b^{\pi}_{c_0} = b^{\nu}_{c_0} = 0$ and $\nu \succsim_{c_0} \pi$, there should be at least one student $s_1 \in \nu(c_0) \setminus \pi(c_0)$ such that s_1 is acceptable for c_0 by Assumption 1 and $c_0 P_{s_1} \pi(s_1)$. We consider two cases regarding $\pi(s_1)$:

Case 1: First, suppose $\pi(s_1) = c_{\emptyset}$. Denote the home college of s_1 by c_1. Hence, $q_{c_1} \geq |\nu(c_1)| > |\pi(c_1)|$ by balancedness of ν and π. By Assumption 1, $\nu(c_1) P^*_{c_1} \pi(c_1)$, and there exists a student $s_2 \in \nu(c_1) \setminus \pi(c_1)$ such that s_2 is acceptable for c_1 and $\nu(s_2) P_{s_2} \pi(s_2)$. Note that, $s_1 \in E_{c_1}$ forms a cycle with $c_{\emptyset}$ before c_1 is removed under 2S-TTC.

Case 2: Next, suppose $\pi(s_1) \in C$. Since $\pi(s_1) \in C$, s_1 is in a cycle that is removed in some round of 2S-TTC. Denote $\pi(s_1)$ by c_1. As $|\nu(c_1)| \geq |\pi(c_1)|$, there exists $s_2 \in \nu(c_1) \setminus \pi(c_1)$, and s_2 is acceptable for c_1 by Assumption 1. We also have $\nu(s_2) P_{s_2} \pi(s_2)$.

We continue with s_2 and $\pi(s_2)$, similarly construct c_2, and then s_3. As we continue, by finiteness, we should encounter the same student $s_k = s_\ell$ for some $k > \ell \geq 1$, that is, we have encountered her before in the construction. Consider the students $s_{\ell+1}, s_{\ell+2}, ..., s_k$. Let $s_{k'}$ be the student who is assigned in the earliest round of 2S-TTC in this list. Suppose $s_{k'}$ is assigned to $\pi(s_{k'})$ in Round $\bar{k}$. By definition, she points to $\pi(s_{k'})$ in Round $\bar{k}$ and $s_{k'} \in E$. However, she prefers $c_{k'-1}$ to her assignment, and she is acceptable for $c_{k'-1}$. Moreover, in Round $\bar{k}$, we know that $c_{k'-1}$ has not been removed yet from the algorithm, because if $c_{k'-1}$ was constructed in Case 1 above, then $q_{c_{k'-1}} > |\pi(c_{k'-1})|$ and $s_{k'-1} \in E_{c_{k'-1}}$ is still not removed, and if $c_{k'-1}$ was constructed in Case 2 above, then $s_{k'-1} \in \pi(c_{k'-1})$ is still not removed. Therefore, $s_{k'}$ should have pointed to $c_{k'-1}$ not $\pi(s_{k'})$ in 2S-TTC in that round. This is a contradiction to ν Pareto dominating π. ■

Proof of Theorem 2. Suppose that there does exist such a mechanism. Denote it by ψ. To show our result, we use several markets that only differ in college preferences.

Case 1: Let $C = \{a, b, c\}$ and $S_a = \{\mathbf{1}, \mathbf{2}\}$, $S_b = \{\mathbf{3}\}$, and $S_c = \{\mathbf{4}\}$. Let $q = e = (2, 1, 1)$. Let $\succsim_S$ be the student preference profile with associated rankings over colleges $P_{\mathbf{1}} : b P_{\mathbf{1}} c P_{\mathbf{1}} c_{\emptyset}$, $P_{\mathbf{2}} : c P_{\mathbf{2}} c_{\emptyset}$, $P_{\mathbf{3}} : a P_{\mathbf{3}} c_{\emptyset}$, and $P_{\mathbf{4}} : a P_{\mathbf{4}} c_{\emptyset}$.[16] Let $\succsim_C$ be the college preference profile with associated rankings over students $P_a : \mathbf{3} P_a \mathbf{4} P_a \emptyset$, $P_b : \mathbf{1} P_b \emptyset$, and $P_c : \mathbf{1} P_c \mathbf{2} P_c \emptyset$.[17] We assume that $\succsim_C$ satisfies Assumption 1. There are two balanced-efficient and individually rational matchings: $\mu_1 = \begin{pmatrix} a & b & c \\ \mathbf{4} & \emptyset & \mathbf{1} \end{pmatrix}$ and $\mu_2 = \begin{pmatrix} a & b & c \\ \{\mathbf{3}, \mathbf{4}\} & \mathbf{1} & \mathbf{2} \end{pmatrix}$.

[14] For the proof of Theorem 9, $|\nu(c)| = |\pi(c)| = q_c$ for all $c \in C$.

[15] For the proof of Theorem 9, since each firm fills all its seats with acceptable workers under 2S-TTC, any balanced matching in which an unacceptable worker is assigned to a firm cannot Pareto dominate 2S-TTC's outcome. Hence, we do not need to consider this case.

[16] In all these rankings, we list only the acceptable colleges.

[17] In all these rankings, we list only the acceptable students.

If ψ selects μ_1, then a can manipulate ψ by submitting $\succsim_a^1$ where $P_a^1 : \mathbf{3}P_a^1\emptyset$ and any acceptable matching under $\succsim$ is preferred to the ones in which $\mathbf{4}$ is assigned to a. Note that, $\succsim_a^1$ satisfies Assumption 1. Then the only individually rational and balanced-efficient matching is $\mu_3 = \begin{pmatrix} a & b & c \\ \mathbf{3} & \mathbf{1} & \emptyset \end{pmatrix}$. Therefore, $\psi[q, e, \succsim] = \mu_2$.

Case 2: We consider the same market with a slight change in a's preferences. Let $\succsim_a^2$ be a' s preferences with associated ranking over students $P_a^2 : \mathbf{4}P_a^2\mathbf{3}P_a^2\emptyset$. We assume that $\succsim_a^2$ satisfies Assumption 1. In this case, μ_1 and μ_2 are the only two balanced-efficient and individually rational matchings.

If ψ selects μ_1, then a can manipulate ψ by submitting $\succsim_a$. Then we will be in Case 1 and μ_2 will be selected, which makes a better off. Therefore, $\psi[q, e, (\succsim_a^2, \succsim_{-a})] = \mu_2$.

Case 3: Now consider the case where colleges report the preferences $\succsim^3$ where $\succsim_a^3 = \succsim_a^2$, $\succsim_b^3 = \succsim_b$, $P_c^3 : \mathbf{1}P_c^3\emptyset$ is the associated ranking with $\succsim_c^3$ and any acceptable matching under $\succsim^3$ is preferred to any matching in which $\mathbf{2}$ is assigned to c under $\succsim_c^3$. Note that, $\succsim_c^3$ satisfies Assumption 1. Then there are two individually rational and balanced-efficient matchings: $\mu_4 = \begin{pmatrix} a & b & c \\ \mathbf{4} & \emptyset & \mathbf{1} \end{pmatrix}$ and $\mu_5 = \begin{pmatrix} a & b & c \\ \mathbf{3} & \mathbf{1} & \emptyset \end{pmatrix}$.

If ψ selects μ_4, then in Case 2 c can manipulate ψ by reporting $\succsim_c^3$. Therefore, $\psi[q, e, \succsim^3] = \mu_5$.

Case 4: Now consider the case where colleges report the following preferences $\succsim^4$ where $\succsim_b^4 = \succsim_b$, $\succsim_c^4 = \succsim_c^3$, $P_a^4 : \mathbf{4}P_a^4\emptyset$ is the associated ranking with $\succsim_a^4$ and any acceptable matching under $\succsim^4$ is preferred to μ_5 under $\succsim_a^4$. Note that, $\succsim_a^4$ satisfies Assumption 1. There is a unique balanced-efficient and individually rational matching: μ_4. In Case 3, a can manipulate ψ by reporting $\succsim_a^4$; then we will be in Case 4 and a will be better off with respect to Case 3 preferences.

Therefore, there does not exist a balanced-efficient, individually rational mechanism that is immune to preference manipulation by colleges. By following the same steps, we can show nonexistence of a mechanism which is acceptable, balanced-efficient, and immune to preference manipulation by colleges. ■

Proof of Theorem 3. For any market $[q, e, \succsim]$, consider the preference relations of each student who ranks as acceptable only those colleges that find her acceptable. If we consider only these preferences as possible preferences to choose from for each student, then 2S-TTC cannot be manipulated by a group of students, as Pápai (2000) showed that TTC is group strategy-proof. In 2S-TTC, observe that students are indifferent among reporting preference relations that rank the colleges finding themselves as acceptable in the same relative order. Therefore, there does not exist a group of students with profitable group manipulation under 2S-TTC.

Thus, 2S-TTC is group strategy-proof for students. ■

The following lemma is used in proving Theorem 4.

Lemma 1 *Let π and $\tilde{\pi}$ be the outcome of 2S-TTC in $\left[q, e, \succsim\right]$ and $\left[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), \succsim\right]$ where $\tilde{q}_c \leq q_c$ and $\tilde{e}_c \leq e_c$ for some $c \in C$, respectively. Then, $M_c^{\tilde{\pi}} \subseteq M_c^{\pi}$, $\tilde{\pi}(c) \subseteq \pi(c)$ and $X_c^{\tilde{\pi}} \subseteq X_c^{\pi}$.*

Proof. If $\tilde{q}_c = q_c$ and $\tilde{e}_c = e_c$, then $\tilde{\pi} = \pi$. Hence, we have three remaining cases to consider.

Case 1: $\tilde{q}_c = q_c$ and $\tilde{e}_c < e_c$. We consider the case in which one more student is certified by c, i.e., $\tilde{e}_c + 1 = e_c$. Denote the student added to the eligible set by s. Let $s' \in S_c$ and $r_c(s') = r_c(s) - 1$. Consider the following variant of the 2S-TTC algorithm for this new market: Suppose there is a cycle consisting of a student $s'' \in S_{c'}$ for some college c' and $c_\emptyset$ in a round and c' has not been removed yet. We remove this cycle if and only if college c' also points to s'' in that round. Otherwise, we keep the cycle in the market to the next round. If $\tilde{q}_c$ students are assigned to c before s is pointed to by c, then c will be removed, and certifying one more student will not affect the set of students exported and imported by c. Now consider the case in which less than $\tilde{q}_c$ students are assigned to c before s is pointed to by c. Denote the intermediate matching that we have just after s' is removed by ν. Since c is removed just after s' is removed in $\left[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), \succsim\right]$, $M_c^{\tilde{\pi}} = M_c^{\nu}$, $\tilde{\pi}(c) = \nu(c)$, and $X_c^{\tilde{\pi}} = X_c^{\nu}$. If s is assigned to a college $c' \in C \setminus c$, c will import one more acceptable student. Denote that matching by μ. Then, we have $M_c^{\tilde{\pi}} = M_c^{\nu} \subset M_c^{\mu}$, $\tilde{\pi}(c) = \nu(c) \subset \mu(c)$, and $X_c^{\tilde{\pi}} = X_c^{\nu} \subset X_c^{\mu}$. If s is assigned to $c_\emptyset$ or c, then c will have the same import and export sets and for the latter case we have $\tilde{\pi}(c) = \nu(c) \subset \mu(c)$. If we keep certifying all $e_c - \tilde{e}_c$ students one at a time, we will have $M_c^{\tilde{\pi}} \subseteq M_c^{\pi}$, $\tilde{\pi}(c) \subseteq \pi(c)$ and $X_c^{\tilde{\pi}} \subseteq X_c^{\pi}$, where π is the outcome of 2S-TTC in $\left[q, e, \succsim\right]$.

Case 2: $\tilde{q}_c < q_c$ and $\tilde{e}_c = e_c$. Let π and ν be the outcomes of 2S-TTC in $\left[q, e, \succsim\right]$ and $\left[(\tilde{q}_c, q_{-c}), e, \succsim\right]$, respectively. If $|\nu(c)| < \tilde{q}_c$ then 2S-TTC will select ν when c reports either $\tilde{q}_c$ or q_c. That is, $\pi = \nu$. If $|\nu(c)| = \tilde{q}_c$ and c's eligibility counter reaches to zero in $\left[(\tilde{q}_c, q_{-c}), e, \succsim\right]$ when it is removed, then it will not make a difference if c reports either $\tilde{q}_c$ or q_c. If $|\nu(c)| = \tilde{q}_c$ and c is removed before all its eligible students are removed in $\left[(\tilde{q}_c, q_{-c}), e, \succsim\right]$, then one more student $s \in S_c$ might be assigned to a college when c reports q_c. As in the previous case, c may import and export at least one more student. At the end, we get $M_c^{\nu} \subseteq M_c^{\pi}$, $\nu(c) \subseteq \pi(c)$ and $X_c^{\nu} \subseteq X_c^{\pi}$.

Case 3: $\tilde{q}_c < q_c$ and $\tilde{e}_c < e_c$. Let μ be the outcome of 2S-TTC in $\left[q, (\tilde{e}_c, e_{-c}), \succsim\right]$. Then, we have $M_c^{\tilde{\pi}} \subseteq M_c^{\mu} \subseteq M_c^{\pi}$, $\tilde{\pi}(c) \subseteq \mu(c) \subseteq \pi(c)$ and $X_c^{\tilde{\pi}} \subseteq X_c^{\mu} \subseteq X_c^{\pi}$, where the first and second subset relations come from invoking Case 1 and Case 2, respectively. ■

Proof of Theorem 4. We prove a stronger version of Theorem 4: Under 2S-TTC, suppose that preference profiles are fixed for colleges such that no college reports an unacceptable student as acceptable in its preference report. In the induced quota-reporting game, under Assumption 1, it is a dominant-strategy equilibrium for all $c \in C$ to certify all their students and to reveal their true admission quotas.

Take a market $\left[q, e, \succsim\right]$ and a college c. Suppose that preference reports are fixed such that c does not report any unacceptable students as acceptable in these reports. Suppose c reports $(\tilde{q}_c, \tilde{e}_c)$ where $\tilde{q}_c \leq q_c$ and $\tilde{e}_c \leq |S_c| = e_c$. In Lemma 1 we have shown that when c reports its admission and eligibility quotas as higher, the set of students assigned to c (weakly) expands. By Assumption 1, reporting $(\tilde{q}_c, \tilde{e}_c)$ is weakly worse than reporting the true admission quota and certifying all students for any profile of other colleges' admission and eligibility quotas (q_{-c}, e_{-c}). ■

Proof of Proposition 2. The 2S-TTC mechanism takes into account only the set of acceptable students based on the submitted preferences of colleges. Hence, for any two different preference profiles with the same set of acceptable students, 2S-TTC selects the same outcome. ■

Proof of Theorem 5. We consider a variant of 2S-TTC in which we select and remove one cycle randomly per round and keep all other cycles intact to the next round. Let $S(k)$ be the set of students in the cycle removed in Round k. To the contrary, suppose the theorem's claim does not hold. Let ψ be the mechanism satisfying all four axioms, and selecting a different matching for some market $\left[q, e, \succsim\right]$. Denote the outcome of 2S-TTC for $\left[q, e, \succsim\right]$ by μ. First note that, $\psi\left[q, e, \succsim\right](s) = \mu(s) = c_\emptyset$ for any ineligible student s.[18] In the rest of the proof, we work with students' preferences over colleges, P_S, instead of $\succsim_S$.

We first prove the following claim:

Claim: If there exists a student in $S(k)$ who prefers her assignment in $\psi\left[q, e, \succsim\right]$ to the one in μ, then there exists another student in $\cup_{k'=1}^{k-1} S(k')$ who prefers her assignment in μ to the one in $\psi\left[q, e, \succsim\right]$.[19]

Proof of Claim: First note that, if for some student s, $\mu(s) \neq \psi\left[q, e, \succsim\right](s)$, then s is an eligible student. We use induction in our proof. Consider the students in $S(1)$. First consider the case in which $|S(1)| = 1$ and the student in $S(1)$ is assigned to $c_\emptyset$. Any college that she prefers to $c_\emptyset$ considers her unacceptable. If she prefers her assignment under ψ to $c_\emptyset$, then she is assigned to a college that considers her unacceptable by ψ. Therefore, ψ is not acceptable. If she prefers $c_\emptyset$ to her assignment under ψ, then ψ is not acceptable.

[18]For the proof of Theorem 9, $\psi\left[q, e, \succsim\right](s) = \mu(s) = c$ for any ineligible worker $s \in S_c$ and $c \in C$.

[19]We take $\cup_{k'=1}^{0} S(k') = \emptyset$.

Then any acceptable mechanism will assign her to $c_\emptyset$. If $|S(1)| > 1$ or $|S(1)| = 1$ and the student in $S(1)$ is assigned to her home college, then each student in $S(1)$ is assigned to the best college that considers her as acceptable, and she prefers her assignment in μ to $c_\emptyset$. If $s \in S(1)$ prefers her assignment in $\psi\big[q, e, \succsim\big]$ to $\mu(s)$, then ψ is not acceptable. Hence, each student in $S(1)$ weakly prefers her assignment in μ. Moreover, by the proof of balanced-efficiency (Part I) of 2S-TTC in Theorem 1, if $\psi\big[q, e, \succsim\big](s) P_s \mu(s)$ for some student $s \in S(2)$, then $\mu(s') P_{s'} \psi\big[q, e, \succsim\big](s')$ for some student $s' \in S(1)$.

In the inductive step, assume that for all Rounds $1, ..., k-1$, for some $k > 1$, the claim is correct. Consider Round k. If there exists a student $s \in S(k)$ such that $c = \psi\big[q, e, \succsim\big](s) P_s \mu(s)$, then either c considers s acceptable and c is removed in Round $\bar{k}$ of 2S-TTC where $\bar{k} < k$, or s is unacceptable for c. In the latter case, ψ is not acceptable. Consider the former case. Note that $c \neq c_\emptyset$.[20] Two cases are possible.

Case 1: First suppose that there exists $s' \in S$ who is assigned to c in μ in Round $k' \leq k - 1$ but not in $\psi\big[q, e, \succsim\big]$. If she prefers c to $\psi\big[q, e, \succsim\big](s')$, then we are done. If she does not, $k' > 1$, and by the inductive step, there exists a student $s'' \in S(k'')$ for some $k'' < k' \leq k - 1$ who prefers $\mu(s'')$ to $\psi\big[q, e, \succsim\big](s'')$.

Case 2: Now suppose $\mu(c) \subset \psi\big[q, e, \succsim\big](c)$. Then, $|\mu(c)| < q_c$ and in Round $\bar{k}$ eligibility quota of c binds under 2S-TTC.[21] By balancedness, there is an eligible student $s'' \in S_c$ who is assigned to $c_\emptyset$ in Round $\tilde{k}$ of 2S-TTC where $1 < \tilde{k} \leq \bar{k} < k$ and $\psi\big[q, e, \succsim\big](s'') \in C$. If she prefers $c_\emptyset$ to $\psi\big[q, e, \succsim\big](s'')$, then we are done. If she does not, by the inductive step, there exists a student $\bar{s} \in S(k'')$ for some $k'' < \tilde{k} \leq k - 1$ who prefers $\mu(\bar{s})$ to $\psi\big[q, e, \succsim\big](\bar{s})$.$\diamond$

Now we are ready to prove the theorem. First note that, we cannot have $\mu \neq \psi\big[q, e \succsim\big]$ and $\mu(s) = \psi\big[q, e \succsim\big](s)$ for all $s \in S(k)$ and $k \leq K$ where K is the last round of 2S-TTC in $\big[q, e \succsim\big]$.

By the Claim and the observation above, as $\mu \neq \psi\big[q, e \succsim\big]$, there exists a student s and some round $k \geq 1$ such that $s \in S(k)$ prefers $\mu(s)$ to $\psi\big[q, e, \succsim\big](s)$, and $\mu(s') = \psi\big[q, e, \succsim\big](s')$ for all $s' \in \cup_{k'=1}^{k-1} S(k')$.

We will construct our proof in three steps. Assign to each round of the 2S-TTC mechanism a counter and set it as Counter$(k') = |S(k')|$ for all rounds $k' \leq K$. In the rest of the proof, we select which cycle to remove in the following manner for the market constructed below while removing only one cycle in every round of the 2S-TTC algorithm: if the cycle removed in Round k of 2S-TTC for $\big[q, e, \succsim\big]$ also exists in Round k of 2S-TTC for this market, then we remove this cycle in that round. Otherwise, we arbitrarily choose

[20] For the proof of Theorem 9, c cannot be the home firm of s.

[21] For the proof of Theorem 9, since $|\mu(c)| = q_c$ for all $c \in C$, this case is not possible.

one cycle.

Step 1: Construct a preference profile $\tilde{\succsim}$ with associated ranking $\tilde{P}$ as follows: Let student $s \in S_c$ rank only $\mu(s)$ as acceptable in $\tilde{P}_s$ and $\tilde{\succsim}_j = \succsim_j$ for all $j \in \big[(C \cup S) \setminus s\big]$. By the execution of the TTC algorithm, 2S-TTC will select μ for $\big[q, e, \tilde{\succsim}\big]$. Since ψ is strategy-proof for students and acceptable, $\psi\big[q, e, \tilde{\succsim}\big](s) = c_\emptyset$.

Then, we check whether the assignments of students in $\cup_{k'=1}^{k-1} S(k')$ are the same in $\psi\big[q, e, \tilde{\succsim}\big]$ and μ. If not, then for some $\tilde{k} < k$, there exists a student $\tilde{s} \in S(\tilde{k})$ preferring $\mu(\tilde{s})$ to $\psi\big[q, e, \tilde{\succsim}\big](\tilde{s})$ and each student in $\cup_{k'=1}^{\tilde{k}-1} S(k')$ gets the same college in μ and $\psi\big[q, e, \tilde{\succsim}\big]$. Then we repeat Step 1 by taking $\succsim := \tilde{\succsim}$, $s := \tilde{s}$, and $k := \tilde{k}$.

This repetition will end by the finiteness of rounds. When all students in $\cup_{k'=1}^{k-1} S(k')$ get the same college in μ, i.e. 2S-TTC outcome in $\big[q, e, \tilde{\succsim}\big]$, and $\psi\big[q, e, \tilde{\succsim}\big]$, then we proceed to Step 2.

Step 2: In Step 1, we have shown that s prefers $\mu(s)$ to $\psi\big[q, e, \tilde{\succsim}\big](s) = c_\emptyset$. Suppose c is the home college of s. Set a new eligibility quota $\tilde{e}_c$ equal to the rank of student s in c's internal priority order, that is, $\tilde{e}_c = r_c(s)$, and let $\tilde{e}_{-c} = e_{-c}$. In $\big[q, \tilde{e}, \tilde{\succsim}\big]$, 2S-TTC assigns all students in $\cup_{k'=1}^{k} S(k')$ to the same college as in μ. $\psi\big[q, \tilde{e}, \tilde{\succsim}\big](s) = c_\emptyset$ since ψ respects internal priorities and we weakly decreased c's eligibility quota. We check whether the assignments of students in $\cup_{k'=1}^{k-1} S(k')$ are the same in both $\psi\big[q, \tilde{e}, \tilde{\succsim}\big]$ and μ. If not, then by the Claim, there should exist $\tilde{s} \in S(\tilde{k})$ preferring $\mu(\tilde{s})$ to $\psi\big[q, \tilde{e}, \tilde{\succsim}\big](\tilde{s})$, and each student in $\cup_{k'=1}^{\tilde{k}-1} S(k')$ gets the same college in μ and $\psi\big[q, \tilde{e}, \tilde{\succsim}\big]$ where $\tilde{k} < k$; then we restart from Step 1 by taking $\succsim := \tilde{\succsim}$, $s := \tilde{s}$, $k := \tilde{k}$, and $e := \tilde{e}$.

Eventually, by the finiteness of the rounds of 2S-TTC and as we reduce the round k in each iteration, we reach the point in our proof such that students in $\cup_{k'=1}^{k-1} S(k')$ get the same college in μ and $\psi\big[q, \tilde{e}, \tilde{\succsim}\big]$.

Observe that s is the last remaining eligible student of c in Round k of 2S-TTC for $\big[q, \tilde{e}, \tilde{\succsim}\big]$ by the choice of $\tilde{e}_c = r_c(s)$. If $|S(k)| = 1$, then $\mu(s)$ is the home college of s. Suppose $|S(k)| > 1$. Since for all $s'' \in \cup_{k'=1}^{k-1} S(k')$, $\mu(s'') = \psi\big[q, \tilde{e}, \tilde{\succsim}\big](s'')$ and $\mu(s) \tilde{P}_s \psi\big[q, \tilde{e}, \tilde{\succsim}\big](s) = c_\emptyset$, some $s' \in S(k) \cap \mu(c)$ will be assigned to a different college in $\psi\big[q, \tilde{e}, \tilde{\succsim}\big]$ than c. Otherwise, ψ is not balanced. As for all $s'' \in \cup_{k'=1}^{k-1} S(k')$, $\mu(s'') = \psi\big[q, \tilde{e}, \tilde{\succsim}\big](s'')$, and s' points to the best available college that finds her acceptable in Round k, $c = \mu(s') \tilde{P}_{s'} \psi\big[q, \tilde{e}, \tilde{\succsim}\big](s')$. We decrease Counter$(k)$ by 1. If Counter$(k) > 0$, then we turn back to Step 1 by taking $\succsim := \tilde{\succsim}$ and $s := s'$; otherwise we continue with Step 3. Note that eventually we will find a Step $\bar{k}$ such that Counter$(\bar{k}) \leq 0$, because we weakly decrease all counters and decrease one counter by 1 in each iteration of Step 2.

Step 3: By the construction above, each $\tilde{s} \in S(k)$ ranks only $\mu(\tilde{s})$ as acceptable in $\tilde{P}_s$ and she is the last certified student by her home college in $\big[q, \tilde{e}, \tilde{\succsim}\big]$. If $|S(k)| = 1$, then $\mu(s)$ is the home college of s and $\psi[q, \tilde{e}, \tilde{\succsim}]$ is Pareto dominated by the matching in which each student other than s is assigned her assignment in $\psi[q, \tilde{e}, \tilde{\succsim}]$ and s is assigned to $\mu(s)$. Therefore, if $|S(k)| = 1$, then $\psi[q, \tilde{e}, \tilde{\succsim}]$ cannot be balanced-efficient. Now, suppose $|S(k)| > 1$. In Step 2, we showed that there exist at least 2 students $s_1(=s$ in Step 2), $s_2(=s'$ in Step 2) in $S(k)$ who are not assigned to $\mu(s_1)$ and $\mu(s_2) = c_1$ ($= c$ in Step 2), respectively, in $\psi[q, \tilde{e}, \tilde{\succsim}]$, where c_1 is the home college of s_1. Then, they are assigned to $c_\emptyset$ in $\psi[q, \tilde{e}, \tilde{\succsim}]$, by the acceptability of ψ. Recall that in 2S-TTC for $[q, \tilde{e}, \tilde{\succsim}]$, each student certified by the home colleges of s_1 and s_2 – colleges c_1 and c_2, respectively – other than s_1 and s_2 is removed in a round earlier than k. Suppose for $s_3 \in S(k)$, $\mu(s_3) = c_2$. Since $\psi[q, \tilde{e}, \tilde{\succsim}](s_2) = c_\emptyset$, for all $\tilde{s} \in \cup_{k'=1}^{k-1} S(k')$, $\psi[q, \tilde{e}, \tilde{\succsim}](\tilde{s}) = \mu(\tilde{s})$ (by Step 2), and ψ is balanced, s_3 cannot be assigned to c_2 in $\psi[q, \tilde{e}, \tilde{\succsim}]$, and hence, $\psi[q, \tilde{e}, \tilde{\succsim}](s_3) = c_\emptyset$. We continue similarly with s_3 and the home college of s_3, say college c_3, eventually showing that for all $\tilde{s} \in S(k)$, $\psi[q, \tilde{e}, \tilde{\succsim}](\tilde{s}) = c_\emptyset$. Recall that students in $S(k)$ had formed a trading cycle in which each student in the cycle was assigned in μ the home college of the next student in the cycle. Thus, $\psi[q, \tilde{e}, \tilde{\succsim}]$ is Pareto dominated by the balanced matching ν obtained as $\nu(\tilde{s}) = \psi[q, \tilde{e}, \tilde{\succsim}](\tilde{s})$ for all $\tilde{s} \in S \setminus S(\mathrm{k})$ and $\nu(\tilde{s}) = \mu(\tilde{s})$ for all $\tilde{s} \in S(k)$. This is because each college in the cycle of Round k gets one acceptable student more and each student in that cycle weakly prefers μ to $\psi[q, \tilde{e}, \tilde{\succsim}]$. This contradicts the balanced-efficiency of ψ. Hence, $\psi[q, e, \succsim] = \mu$, i.e., ψ is equivalent to 2S-TTC. ■

Proof of Theorem 6. Let ψ satisfy all conditions and be strategy-proof for students. Then, consider the following market. There are 3 colleges $C = \{a, b, c\}$ with $q = e = (2, 1, 1)$. Let $S_a = \{\mathbf{1}, \mathbf{2}\}$, $S_b = \{\mathbf{3}\}$, $S_c = \{\mathbf{4}\}$, and each student be acceptable to each college and the college preference profile satisfy Assumption 1. The internal priority order of a and student preference profiles are given as: $\mathbf{1} \rhd_a \mathbf{2}$, $bP_{\mathbf{1}}c_\emptyset$, $cP_{\mathbf{2}}c_\emptyset$, $aP_{\mathbf{3}}c_\emptyset$, $bP_{\mathbf{4}}aP_{\mathbf{4}}c_\emptyset$.[22]

2S-TTC selects $\mu = \left(\begin{array}{ccc} a & b & c \\ \{\mathbf{3}, \mathbf{4}\} & \mathbf{1} & \mathbf{2} \end{array}\right)$. ψ will also select μ, since any other matching in which all students are assigned is individually irrational (and unacceptable).

If student $\mathbf{4}$ reports $\succsim'_{\mathbf{4}}$ with associated ranking $P'_{\mathbf{4}} : bP'_{\mathbf{4}}c_\emptyset P'_{\mathbf{4}}a$ then 2S-TTC will select $\mu' = \left(\begin{array}{ccc} a & b & c \\ \mathbf{3} & \mathbf{1} & \emptyset \end{array}\right)$. The only balanced and individually rational (acceptable) matching in which more than two students are assigned is $\mu'' = \left(\begin{array}{ccc} a & b & c \\ \mathbf{3} & \mathbf{4} & \mathbf{2} \end{array}\right)$. Therefore, the outcome of ψ when $\mathbf{4}$ reports $\succsim'_{\mathbf{4}}$ is μ''. Hence, $\mathbf{4}$ can manipulate ψ. ■

Proof of Theorem 7. We first prove the strategy-proofness of 2S-TTTC for students.

[22] In all these rankings, we list only the acceptable colleges.

Consider a tuition exchange market $\left[q, e, \succsim\right]$ and tolerance profile $(\ell_c, u_c)_{c\in C}$. We use a variation of 2S-TTTC in which in each round only the students who are pointed to by their home colleges can point to a college in C. Let μ be the matching selected by 2S-TTTC in $\left[q, e, \succsim\right]$. First note that, any student who is assigned to $c_\emptyset$ after the termination of the algorithm has never been pointed by her home college and she cannot change her match by misreporting. Let $k > 0$ be the first round that we cannot locate a cycle. Note that in Round k either there exists a chain, which may or may not respect the tolerance profile, or the algorithm terminates. Student s assigned in Round $k' < k$ (under truth telling) cannot affect the cycles that formed in earlier rounds. Before Round k', all colleges, which consider s acceptable and s prefers to $\mu(s)$, should have been removed or become non-importing. If s forms a cycle by misreporting in Round $k'' < k'$, then she should have pointed to a worse option than $\mu(s)$. Therefore, student s cannot get a better match by misreporting.

Now consider Round k. If Round k is the termination round, then we are done. Otherwise, firstly assume that we have a chain that respects the tolerance profile. Any active student in Round k cannot affect the cycles that formed in earlier rounds. Then consider the student pointed to by the tail college of the chain. This student will be assigned in this round no matter which college she points to. Therefore, it is in her best interest to be truthful so that she points to her most preferred importing college, which considers her acceptable, among the available ones. This argument is also true for the other students in the chain.

Now consider the case where we do not have a chain that respects the tolerance profile. That is, each exporting college c has already a balance of ℓ_c. Then we will remove all the non-exporting colleges; 2S-TTTC reduces to the 2S-TTC mechanism. It is easy to see that we will not have chains respecting the tolerance profile in the future rounds, either. Moreover, the remaining students cannot prevent the removal of non-exporting colleges in this round by changing their preferences.

For the remaining rounds, we can show that no student can gain from misreporting by following the same reasoning.

Next we prove 2S-TTTC's outcome cannot be Pareto dominated by an acceptable matching ν that satisfies the tolerance profile. Note that, $\nu(s) = \emptyset$ for any ineligible student s. Denote the outcome of the 2S-TTTC mechanism with μ. We first consider the students who are assigned before the termination of the algorithm. Let $k \geq 1$ be the first round that we cannot locate a cycle. We consider the variant that we described before. As described above, either k is the termination round or there exists a chain in Round k. In the first round, each student is pointing to her favorite among available importing colleges, which consider her acceptable, and $c_\emptyset$. If a student is assigned in this round, then she should get the same college in ν. Now consider students assigned in Round $k' < k$ when $k > 1$. All the colleges that a student prefers to her assignment and consider her acceptable should have been removed or become non-importing in an earlier round. We cannot make this student better off by assigning her to a college that considers her acceptable without hurting another student assigned in an earlier round or violating the tolerance conditions or feasibility constraints.

If k is the termination round, we are done. Otherwise, we consider the students assigned in Round k. First, consider the case where there exists a chain not violating the tolerance conditions. All students in that chain are assigned to importing colleges that they prefer most among the available ones considering them acceptable. They cannot be made better off without making some students assigned in the earlier rounds worse off or violating the tolerance conditions or feasibility constraints. If there does not exist a chain respecting the tolerance profile, then 2S-TTTC reduces to the 2S-TTC mechanism. After this round, assigning a student to a college by not following a trade in an encountered cycle will violate the tolerance conditions or feasibility constraints.

For the remaining rounds, by following the same reasoning, we can show that no student can be made better off without either hurting another student or violating the tolerance conditions or feasibility constraints.

Moreover, if we assign the students, who were assigned to $c_\emptyset$ after the termination of the algorithm, to some college in ν, then either the tolerance conditions or feasibility constraints are violated. Hence, no college can be made better off without harming another agent or violating the tolerance conditions or feasibility constraints. ∎

Proof of Theorem 8. We refer to the proof of Theorem 1. We replace the word "cycle" with "cycle or chain" throughout the proof and the proof holds. ∎

Proof of Theorem 9. In any market, under 2S-TTC when a firm is removed its eligibility counter reaches zero. Hence, each eligible worker will be pointed by her home firm at some round and she will be assigned to a firm weakly better than her home firm. Moreover, each ineligible worker is assigned to her home firm. Hence, each worker is assigned to an acceptable firm. Moreover, each firm is only pointed by the workers it considers as acceptable. That is, for any problem, 2S-TTC selects a matching which is acceptable. Moreover, 2S-TTC is individually rational since each agent is matched with acceptable agents and a firm c cannot block 2S-TTC's outcome since all other firms fill their seats in any matching. The part of the proof of Theorem 1 for respecting internal priorities and balanced-efficiency hold. Note that, as all matchings are balanced in this domain, balanced-efficiency and Pareto efficiency are equivalent concepts.

Since we can run 2S-TTC initially assigning all ineligible workers to their home firms, the proof of Theorem 3 implies the worker-strategy-proofness of 2S-TTC.

The proof of Theorem 5 for uniqueness holds with a slight change. First note that any Pareto efficient, student-strategy-proof and acceptable mechanism assigns workers to either their home firms or better firms that consider them acceptable. In the uniqueness part of the proof (i.e. Theorem 5's proof adopted for 2S-TTC being the only mechanism satisfying Pareto efficiency, student-strategy-proofness, acceptability, and respect for internal priorities in the temporary worker exchange model), while updating worker s's preferences in Step 1, we do it as follows: rank $\mu(s)$ and her home firm as only acceptable firms in the correct order of her true preferences. And then at the end of Step 1, she will be assigned to her home firm under ψ. Since ψ respects internal priorities and is acceptable, student-strategy-proof, and balanced-efficient, s will remain at her home firm in Step 2. When we reach Step 3, we will have a set of workers who are assigned to their home firms by ψ; however, a trading cycle between them would improve total welfare without violating balancedness or feasibility.

Immunity to Preference Manipulation by Colleges: Recall that in any matching balancedness is satisfied and firms fill their admission quotas. Hence, under Assumption 2, firms are indifferent between any acceptable matching. Since the 2S-TTC mechanism selects an acceptable matching when firms report truthfully, firms cannot be better off by manipulating their preferences over the matchings and reporting quotas different from their true quotas.

Stability: Consider an arbitrary market $\left[q, e, \succsim\right]$. Denote the outcome of 2S-TTC by μ. Recall that $q_c = |S_c|$ for all $c \in C$, all workers consider their current firms acceptable, all firms consider their current workers acceptable, and workers who are not certified remain at their current firms. Hence, $|\mu(c)| = q_c$ for all $c \in C$. Since in μ all firms' quotas are filled, μ is nonwasteful. Note that, any mutual deviation of worker-firm pair needs to end up with a (balanced) matching. Since all employees in $\mu(c)$ are acceptable, replacing one of the employees in $\mu(c)$ with another one in $S \setminus \mu(c)$ cannot make c better off. Hence, μ cannot be blocked by a worker-firm pair. ■

Appendix C Tuition-Exchange Programs

We first explain why tuition-exchange programs exist in the first place because some colleges choose to subsidize faculty directly instead of participating in tuition-exchange programs. Although this may create flexibility for the students, any direct compensation over \$5,250 is taxable income, whereas a tuition-exchange scholarship is not.[23] Tuition exchange is not considered to be an income transfer.[24] Moreover, colleges may not want to switch to such direct-compensation programs from a cost-saving perspective, regardless of the tax benefit to the faculty member. We present a simple back-of-the envelope calculation to demonstrate these cost savings. There are more than 1,800 4-year colleges in the US and at most half of them have membership to at least one tuition-exchange program. Suppose n students are given tuition exchange/remission scholarships a year. Instead, if a college finances the tuition of a faculty member's child through direct cash compensation, then all tuition exchange colleges will have to pay $\$n\bar{T}$, where $\overline{T}$ is the average full tuition cost of colleges. However, assuming that average qualities and sizes of colleges with and without tuition scholarship are the same, only half of these students will attend a tuition exchange college in return; so the colleges will only get back $\$\frac{n\overline{T}}{2}$. The remaining $\frac{n}{2}$ slots will be filled with regular students. Regular students on average pay about half of the tuition thanks to other financial aid programs. For example, 2012 Tuition Discounting Study of the National Association of College and University Business Officers report that incoming freshmen pay on average 56% of full tuition at a private university. Thus, they will only pay $\$\frac{n\overline{T}}{4}$ to tuition exchange colleges. As tuition exchange scholarships constitute a very small portion of college admissions, this calculation assumes that average tuition payment would not change by establishment of direct cash compensation instead of tuition exchange. Thus, as a result, the colleges will lose in total about $\$\frac{n\overline{T}}{4}$, which corresponds to one fourth of average full tuition per student. Thus, the total per-student-savings for the faculty member and the college is more than half of tuition payment - assuming one third of the direct compensation is paid in income tax at the margin by the parent.

The Tuition Exchange Inc (TTEI): In addition to information provided in the Section 2, here we give more detail. TTEI is a reciprocal scholarship program for children (and other family members) of faculty and staff employed at more than 600 colleges. Member colleges are spread over 47 states and the District of Columbia. Both research universities and liberal arts colleges are members. *US News and World Report* lists 38 member colleges in the best 200 research universities and 46 member colleges in the best 100 liberal arts colleges.

In TTEI, every participating institution determines the number of outgoing students it can certify, as well as how many TTEI awards it will grant to incoming students each

[23]See https://www.irs.gov/newsroom/tax-benefits-for-education-information-center reached on Feb 18, 2018.

[24]In particular, it is considered a scholarship, and it is not taxable. See https://www.irs.gov/publications/p970 reached on Feb 18, 2018.

year. Then each faculty member submits the TTEI application to the registration office of their college. If the number of applicants is greater than the number of students that the college is willing to certify, then the college decides whom to certify based on years of service or some other criterion (internal priority order).

Each student who is certified eligible submits a list of colleges to the liaison office of her home institution. Each liaison office sends a copy of the TTEI "Certificate of Eligibility" to the TTEI liaison officer at the participating colleges and universities listed by the eligible dependents. Certification only means that the student is eligible for a TTEI award; it is not a guarantee of an award. The eligible student must apply for admission to the college(s) in which she is interested, following each institution's application procedures and deadlines. After admission decisions have been made, the admissions offices or TTEI liaisons at her listed institutions inform her whether she will be offered a TTEI award. TTEI scholarships are competitive, and some eligible applicants may not receive them. That is, the sponsoring institution cannot guarantee that an "export" candidate, regardless of qualifications, will receive a TTEI scholarship. Institutions choose their scholarship recipients ("imports") based on the applicants' academic profiles.

To collect anecdotal evidence on how much faculty members value the tuition-exchange benefit, we also conducted an IRB-approved e-mail-delivered online survey in 21 tuition-exchange colleges (all TTEI members and possibly members of other tuition exchange programs) using Qualtrics e-mail survey software. Our respondent pool is composed of 153 faculty members (with a 7.5% to 15% response rate). In this pool, there are 47, 56, and 50 assistant, associate, and full professors, respectively. 17% of the respondents have no child. In order to understand whether tuition-exchange benefits attract faculty members, we ask how important of a role their college's membership in a tuition-exchange program played their acceptance of their offer. According to 19%/57% of the respondents, the tuition-exchange benefit was extremely important/important in their acceptance decision, respectively. Moreover, according to 23%/62% of the respondents with children, the tuition-exchange benefit played an extremely important/important role in their acceptance decision, respectively. In order to understand the value of the tuition-exchange benefit for faculty, we asked how much annual income they would give up in order to keep their tuition-exchange benefit. When we consider all respondents, the average annual value of the tuition-exchange benefit is $7,636 each year in today's dollars per faculty member (for the ones with one or more child currently, it is only slightly higher, $8,516).
The Council of Independent Colleges Tuition Exchange Program (CIC-TEP): CIC-TEP is composed of almost 500 colleges. All full-time employees of the member colleges and their dependents can benefit from this program. Each college certifies its own

employees eligible based on its own rules. Each member college is required to accept at least three exchange students per year. There is no limitation on the number of exported students. Each certified student also applies for admission directly to the member colleges of her choice. Certified students must be admitted by the host college in order to be considered for the tuition exchange scholarship. Each year more than 1,500 students benefit from this program.

Catholic College Cooperative Tuition Exchange (CCCTE): CCCTE is composed of 70 member colleges. Each member college certifies its employees as eligible based on its own rules. Students must be admitted by the host college before applying for the tuition exchange scholarship. Admission does not guarantee the scholarship. Each member college can have at most five more import students than its exports. The number of exported students is not limited.

Great Lakes Colleges' Association (GLCA): GLCA is composed of thirteen liberal arts colleges in Pennsylvania, Michigan, Ohio, and Indiana. Each member college determines the eligibility of its employees based on its own rules. All other policies are determined by the host colleges. Each accepted student pays a fee equal to 15% of the GLCA mean tuition. The remaining tuition is paid by the home college.

Associated Colleges of the Midwest (ACM): ACM is composed of fourteen liberal arts colleges in Wisconsin, Minnesota, Iowa, Illinois, and Colorado. Eligibility of the students is determined based on the home college rules. Each host college compensates 50% tuition to all imported students. The remaining portion of the tuition is paid by the home college and the student.

Faculty and Staff Children Exchange Program (FACHEX): FACHEX is composed of 28 Jesuit colleges. Each student first applies to be admitted by the host college. Admission to the host college does not guarantee receiving tuition exchange scholarship.

Council for Christian Colleges and Universities Tuition-Waiver Exchange Program (CCCU-TWEP): CCCU-TWEP is composed of 100 colleges. Each member college must accept at least one exchange student. In order to receive tuition exchange scholarship, each student needs to be admitted by the host college.

Appendix D Temporary Worker-Exchange Programs

D.1 Teacher Exchange

The Fulbright Teacher Exchange Program, established by an act of the US Congress in 1946, provides opportunities to school teachers in the US to participate in a direct exchange of positions with teachers from countries, including the Greece, Finland, Netherlands, India, Mexico, and the UK. Matching procedure is arranged by the Fulbright program staff, and each candidate and each school must be approved before the matchings are finalized.

The Commonwealth Teacher Exchange Programme (CTEP) was founded by the League for the Exchange of Commonwealth Teachers more than 100 years ago. Participant teachers exchange their jobs and homes with each other usually for a year, and they stay employed by their own school. Countries participating to this program are Australia, Canada, and the UK. More than 40,000 teachers have benefited from the CTEP. Principals have the right to veto any proposed exchange they think will not be appropriate for their school.

The Educator Exchange Program is organized by the Canadian Education Exchange Foundation. The program includes reciprocal interprovincial and international exchanges. The international destinations are Australia, Denmark, France, Germany, Switzerland, the UK, and Colorado, the US.

The Manitoba Teacher Exchange enables teachers in Manitoba to exchange their positions with teachers in Australia, the UK, the US, Germany, and other Canadian provinces. Once a potential match is found, the incoming teacher's information is sent to the Manitoba applicant, the principal of the school, and the employing authority. Acceptance of all these teachers is required for the completion of the exchange.

In the **Saskatchewan Teacher Exchange**, public school teachers with at least five years of experience can apply for exchange positions with teachers in the UK, the US, and Germany. Potential exchange candidates are determined based on similar teaching assignments and they are sent to applicant's director of education. If the potential exchange candidates are considered acceptable, then the applicant will consider the candidate. The exchange is finalized once the applicant accepts it.

The Northern Territory Teacher Exchange Program is a reciprocal program in which teachers in Northern Australia exchange positions with teachers from the UK, Canada, the US, New Zealand, and other Australian states. When a potential match is found for an applicant, the applicant and her school principal decide whether to accept or reject the proposal. The match is finalized when both sides accept it.

The Western Australian Teacher Exchange is a reciprocal program. The match is finalized after the approval of the principals of both sides.

The Rural Teacher Exchange is a reciprocal program which gives opportunity to teachers in more than 800 rural schools in New South Wales to exchange their positions. Exchanges are selected via centralized mechanism. However, if a teacher can find a possible exchange counterpart, then they can exchange their positions before entering the central mechanism.

D.2 Clinical Exchange

In **the International Clinical Exchange Program,** medical students exchange positions with other medical students from other countries. The program is run by the International Federation of Medical Students Association. Every year, approximately 13,000 students exchange their positions. The exchanges are done bilaterally. In a county, the exact number of available positions available for another country is determined by the number of contracts signed between both countries.

The MICEFA Medical Program has enabled medical students in France and the US to exchange their positions for one to two months for 30 years. Students are exchanged on a one-to-one basis and each exchange students pays tuition to her home institute.

D.3 Student Exchange

The National Student Exchange (**NSE**), established in 1968, is composed of nearly 200 colleges from the US, Canada, Guam, Puerto Rico, and the US Virgin Islands. More than 105,000 undergraduate students have exchanged their colleges through NSE. Exchange students pay either the in-state tuition of their host college or the normal tuition of their home college.

The University of California Reciprocal Exchange Program enables the students of the University of California system to study in more than 120 universities from 33 countries. Around 4,000 students benefit from this program annually. Exchange students are selected by their home universities. This is a reciprocal exchange program and it aims to balance the costs and benefits of import and export students for each university.

The University Mobility in Asia and the Pacific Exchange Program (**UMAPEP**), established in 1993, is a student exchange program between 500 universities in 34 Asia-Pacific countries. UMAPEP involves two programs: a bilateral exchange program and a multilateral exchange program. In the bilateral exchange program, home colleges select the exchange students and exchanges are done through bilateral agreements signed between the member colleges. In the multilateral exchange program, host universities select the incoming exchange students.

The International Student Exchange (**ISE**), founded in 1979, is a reciprocal program. Around 40,000 students from 45 countries have benefited from ISE. Each exchange student pays tuition to her home college.

The Erasmus Student Exchange Program is a leading exchange program between the universities in Europe. Close to 3 million students have participated since it started in 1987. The number of students benefiting from the program is increasing each year; in 2011, more than 230,000 students attended a college in another member country as an exchange student. The number of member colleges is more than 4,000. Each college needs to sign bilateral agreements with the other member institutions. In particular, the student exchanges are done between the member universities that have signed a bilateral contract with each other. The bilateral agreement includes information about the number of students who will be exchanged between the two universities in a given period. The selection process of the exchange students is mostly done as follows. The maximum number of students that can be exported to a partner university is determined based on the bilateral agreement with that partner and the number of students who have been exported since the agreement was signed. The students submit their list of preferences over the partner universities to their home university. Each university ranks its own students based on predetermined criteria, e.g., GPA and seniority. Based on the ranking, a serial dictatorship mechanism is applied to place students in the available slots. Finally, the list of students who received slots at the partner universities is sent to the partners. The partner universities typically accept all the students on the list. An exchange student pays her tuition to her own college, not the one importing her.

There are huge imbalances between the number of students exported and imported by each country. Moreover, countries with high positive balances are not often willing to match the quota requests of the net-exporter countries. This precautionary behavior may lead to inefficiencies as in tuition-exchange markets.

D.4 Scientific Exchange

The Mevlana Exchange Program aims to exchange academic staff between Turkish universities and universities in other countries. Turkish public universities are governed by the Turkish Higher Education Council and professors are public servants. Therefore, the part of the exchange that is among public universities can be seen as a staff-exchange program, while the exchange among public and private Turkish universities and foreign universities can be seen as a worker-exchange program. Any country can join this program. In 2013, around 1,000 faculty members benefited from this program.

Appendix E Proofs of Appendix A

Proof of Proposition 3. We prove existence by showing that for any tuition-exchange market there exists an associated college admission market and the set of stable matchings are the same under both markets. Under Assumption 3, we fix a tuition exchange market $[q, e, \succsim]$. Let E be the set of eligible students. We first introduce an associated college admissions market, i.e., a Gale-Shapley (1962) two-sided many-to-one matching market, $[S, C, q, P_S, \overline{P}_C]$, where the set of students is S; the set of colleges is C; the quota vector of colleges for admissions is q; the preference profile of students over colleges is P_S, which are all the same entities imported from the tuition exchange market; and the preference profile of colleges over the set of students is $\overline{P}_C$, which we construct as follows: for all $T \subset S$ with $|T| < q_c$ and $i, j \in E \backslash T$, (i) $i \; P_c \; j \implies (T \cup i) \; \overline{P}_c \; (T \cup j)$, (ii) $i \; P_c \; \emptyset \iff (T \cup i) \; \overline{P}_c \; T$, and (iii) $T \; \overline{P}_c \; (T \cup k)$ and $k \; P_c \; \ell \implies (T \cup i) \; \overline{P}_c \; (T \cup k) \; \overline{P}_c \; (T \cup \ell)$ for all $k, \ell \in S \setminus E$. Note that, $\overline{P}_C$ is responsive up to quota. We fix C and S and represent such a college admission market as $[q, P_S, \overline{P}_C]$. In this college admissions market, a *matching* $\overline{\mu}$ is a correspondence $\overline{\mu} : C \cup S \twoheadrightarrow C \cup S \cup c_\emptyset$ such that (1) $\overline{\mu}(c) \subseteq S$ where $|\overline{\mu}(c)| \leq q_c$ for all $c \in C$, (2) $\overline{\mu}(s) \in C \cup c_\emptyset$ where $|\overline{\mu}(s)| = 1$ for all $s \in S$, and (3) $s \in \overline{\mu}(c) \iff \overline{\mu}(s) = c$ for all $c \in C$ and $s \in S$. A matching $\overline{\mu}$ is *individually rational* if $\overline{\mu}(s) \; R_s \; c_\emptyset$ for all $s \in S$, and, for all $s \in \overline{\mu}(c)$, we have $s \overline{P}_c \emptyset$ for all $c \in C$. A matching $\overline{\mu}$ is *nonwasteful* if there does not exist any $(c, s) \in C \times S$ such that (1) $c \; P_s \; \overline{\mu}(s)$, (2) $|\overline{\mu}(c)| < q_c$, and (3) $s \overline{P}_c \emptyset$. A matching $\overline{\mu}$ is *blocked by a pair* $(c, s) \in C \times S$ if $c \; P_s \; \overline{\mu}(s)$, and there exists $s' \in \overline{\mu}(c)$ such that $s \; \overline{P}_c \; s'$. A matching $\overline{\mu}$ is *stable* in a college admission market if it is individually rational, nonwasteful, and not blocked by any pair.

By our construction $\overline{P}_C$ is responsive up to quota; hence there exists at least one stable matching for $[q, P_S, \overline{P}_C]$ (see Gale and Shapley, 1962; Roth, 1985). Let $\overline{\mu}$ be a stable matching for $[q, P_S, \overline{P}_C]$. We first show that $\overline{\mu}$ is also a matching for $[q, e, \succsim]$. Due to individual rationality, $\overline{\mu}(s) = c_\emptyset$ for all $s \notin E$. By the definition of a matching in a college admission market, other parts of the definition of a matching in a tuition exchange market hold. Hence, $\overline{\mu}$ is a matching for $[q, e, \succsim]$.

Now, we show that $\overline{\mu}$ is stable for $[q, e, \succsim]$. Due to individually rationality of $\overline{\mu}$ in the college admission market, $\overline{\mu}(s) R_s c_\emptyset$ and $s P_c \emptyset$ for all $s \in \overline{\mu}(c)$ and $c \in C$. By Assumption 3 and the definition of individual rationality in the tuition-exchange market, $\overline{\mu}$ is individually rational in $[q, e, \succsim]$. Whenever there exists $s \in S$ such that $c P_s \overline{\mu}(s)$, then either $s \in S \setminus E$ or $\overline{\mu} \succ_c \mu'$ for all $\mu' \in \mathcal{M}$, where $s \in \mu'(c) \subseteq \overline{\mu}(c) \cup s$ and $\overline{\mu}(s') = \mu'(s')$ for all $s' \in S \setminus (\overline{\mu}(c) \cup s)$. This follows from the definition of stability and construction of the college preferences in the associated college admission market and Assumption 3. Hence, $\overline{\mu}$ is stable for $[q, e, \succsim]$.

Finally, we show that if a matching is not stable for $[q, P_S, \overline{P}_C]$, then it is either not a matching or unstable for $[q, e, \succsim]$. Note that any matching for $[q, e, \succsim]$ is also a well-defined matching for $[q, P_S, \overline{P}_C]$. Hence, it suffices to show that any matching μ for $[q, e, \succsim]$ that is not stable for $[q, P_S, \overline{P}_C]$ fails to be stable for $[q, e, \succsim]$.[25] If μ is blocked by an agent in $[q, P_S, \overline{P}_C]$, then by our assumption on the preferences it is also blocked by the same agent in $[q, e, \succsim]$. If μ is wasteful for $[q, P_S, \overline{P}_C]$, then there exists a college-student pair (c, s) such that $|\mu(c)| < q_c$, $s \in E$, $cP_s\mu(s)$, $s\overline{P}_c\emptyset$ and her addition to the set of students admitted by c in μ and keeping all other students assignment the same is both preferred by c and herself in $[q, e, \succsim]$. Similarly, if (c, s) is a blocking pair in $[q, P_S, \overline{P}_C]$ then by our preference construction and stability definition (c, s) is a blocking pair in $[q, e, \succsim]$. Thus, if μ is a stable matching for $[q, e, \succsim]$, it is also stable for $[q, P_S, \overline{P}_C]$.

Hence, the set of stable matchings for $[q, e, \succsim]$ and the set of stable matchings for $[q, P_S, \overline{P}_C]$ are the same. ■

Proof of Proposition 5. Under Assumption 3, we fix a market $[q, e, \succsim]$. The case in which we have a unique stable matching for $[q, e, \succsim]$ is trivial. Hence, we consider the case in which there are at least two stable matchings. Let ν and μ be any two stable matchings for $[q, e, \succsim]$. By the proof of Proposition 3, ν and μ are also stable for the associated college admission market $[q, P_S, \overline{P}_C]$. Let S^ν and S^μ be the set of students assigned to a college in ν and μ, respectively. Due to Assumption 3 Part 3 and individual rationality, $M_c^\mu = \mu(c)$, $M_c^\nu = \nu(c)$ for all $c \in C$. In the rural hospital theorem (Roth, 1986) it is shown that the number of students assigned to a college is the same in all stable matchings, $|\nu(c)| = |\mu(c)|$ for each $c \in C$. Moreover, the set of students assigned to a college is the same in all stable matchings, i.e., $S^\nu = S^\mu$. Since $X_c^\mu = S^\mu \cap S_c$, $X_c^\nu = S^\nu \cap S_c$, and $S^\nu = S^\mu$, we have $X_c^\mu = X_c^\nu$. Then, $b_c^\mu = |\mu(c)| - |S^\mu \cap S_c| = |\nu(c)| - |S^\nu \cap S_c| = b_c^\nu$ for all $c \in C$. ■

We first state and prove the following Lemma, which is used in proving Proposition 6 and Theorem 11.

Lemma 2 *Under Assumption 3, let $\hat{\pi}$ be a stable matching for $[\hat{q}, \hat{e}, \succsim]$ and $\tilde{\pi}$ be a stable matching for $[(\tilde{q}_c, \hat{q}_{-c}), (\tilde{e}_c, \hat{e}_{-c}), \succsim]$ where $\tilde{e}_c = \hat{e}_c + 1$, and $\tilde{q}_c = \hat{q}_c$ if $|\hat{\pi}(c)| = \hat{q}_c$ and $\tilde{q}_c \geq \hat{q}_c$ otherwise. Then we have $b_c^{\tilde{\pi}} \in \{b_c^{\hat{\pi}} - 1, b_c^{\hat{\pi}}\}$ and $b_{c'}^{\tilde{\pi}} \in \{b_{c'}^{\hat{\pi}}, b_{c'}^{\hat{\pi}} + 1\}$ for all $c' \in C \setminus c$.*

Proof. Let E be the set of eligible students in $[\hat{q}, \hat{e}, \succsim]$. Denote the newly certified student of c by i in $[(\tilde{q}_c, \hat{q}_{-c}), (\tilde{e}_c, \hat{e}_{-c}), \succsim]$. The net balance of each college is the same

[25] This observation implies that, there does not exist a stable matching for $[q, e, \succsim]$ that is not stable for $[q, P_S, \overline{P}_C]$.

at every stable matching by Proposition 5. Moreover, $\hat{\pi}$ is stable for the associated college admissions market $\left[\hat{q}, P_S, \overline{P}_C\right]$ by the proof of Proposition 3. Thus, without loss of generality, we assume $\hat{\pi}$ to be the outcome of the (student-proposing) DA algorithm for $\left[\hat{q}, P_S, \overline{P}_C\right]$.

First, consider the market $\left[(\tilde{q}_c, \hat{q}_{-c}), \hat{e}_c, \succsim\right]$. Let $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}'_C\right]$ be the associated college admissions market. Note that, under both $\overline{P}'_C$ and $\overline{P}_C$ the rankings over the individual students are the same for all colleges. If $|\hat{\pi}(c)| < \hat{q}_c$, then adding new seats to an underdemanded college will not change the set of students assigned to c, and DA selects the same outcome in $\left[\hat{q}, P_S, \overline{P}_C\right]$ and $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}'_C\right]$. If $|\hat{\pi}(c)| = \hat{q}_c$, $\tilde{q}_c = \hat{q}_c$ by assumption. Hence, DA selects the same outcome for $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}'_C\right]$ and $\left[\hat{q}, P_S, \overline{P}_C\right]$.

Denote the associated college admissions market of $\left[(\tilde{q}_c, \hat{q}_{-c}), (\tilde{e}_c, \hat{e}_{-c}), \succsim\right]$ by $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}''_C\right]$. Note that the preference profile of the colleges change in the related college admission market since we change the set of eligible students. However, the rankings over the individual students in E under both $\overline{P}_C$ and $\overline{P}''_C$ for all colleges are the same. We will apply the sequential DA algorithm introduced by McVitie and Wilson (1971) for $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}''_C\right]$, where the newly certified student i will be considered at the end. Let $\tilde{\pi}$ be the outcome of DA for $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}''_C\right]$.

Let $C_<$ be the set of colleges that could not fill all their seats, and $C_=$ be the set of colleges that did, in $\hat{\pi}$. Formally, $C_< = \{c \in C \ : \ |\hat{\pi}(c)| < \hat{q}_c\}$ and $C_= = \{c \in C \ : \ |\hat{\pi}(c)| = \hat{q}_c\}$. Now, when it is the turn of i to apply in the sequential version of the student-proposing DA, the current tentative matching is $\hat{\pi}$. After i starts making applications in the algorithm, let $\tilde{c}$ be the first option that does not reject i. Since $\emptyset P_c i$, $c \neq \tilde{c}$, i.e., $\tilde{c}$ is not i's home college.

In the rest of the proof, as we run the sequential DA, we run the following cases iteratively, starting with student i:

1. If $\tilde{c} = c_\emptyset$, then the algorithm terminates; $b^{\tilde{\pi}} = b^{\hat{\pi}}$.

2. If $\tilde{c} \in C_<$, then i will be assigned to $\tilde{c}$ and the algorithm terminates; $b^{\tilde{\pi}}_c = b^{\hat{\pi}}_c - 1$, $b^{\tilde{\pi}}_{\tilde{c}} = b^{\hat{\pi}}_{\tilde{c}} + 1$, and $b^{\tilde{\pi}}_{c'} = b^{\hat{\pi}}_{c'}$ for all $c' \in C \setminus \{c, \tilde{c}\}$.

3. If $\tilde{c} \in C_=$, then student $\tilde{i}$ who is the least preferred student among the ones in $\hat{\pi}(\tilde{c})$ is rejected in favor of i. We consider two cases:

3.a. Case $\tilde{i} \in S_c$: The net balance of no college will change from the beginning, and we continue from the beginning above, again using student $\tilde{i}$ instead of i.

3.b. Case $\tilde{i} \notin S_c$: The instantaneous balance of c will deteriorate by 1 as i is tentatively accepted. Now, it is $\tilde{i}$'s turn in the sequential DA to make offers. In this series of offers, suppose option that does not reject student $\tilde{i}$ is $\tilde{\tilde{c}}$. Denote the home college of $\tilde{i}$ by c' (note that $c' \neq \tilde{c}$).

3.b.i. If $\tilde{\tilde{c}} \in c_\emptyset \cup C_<$, then the algorithm will terminate, and $b_c^{\tilde{\pi}} \in \{b_c^{\hat{\pi}} - 1, b_c^{\hat{\pi}}\}$, $b_{\bar{c}}^{\tilde{\pi}} \in \{b_{\bar{c}}^{\hat{\pi}}, b_{\bar{c}}^{\hat{\pi}} + 1\}$ for all $\bar{c} \in C \setminus c$.

3.b.ii. If $\tilde{\tilde{c}} \in C_=$, then the least preferred student held by $\tilde{\tilde{c}}$ will be rejected in favor of $\tilde{i}$. Let this student be $\tilde{\tilde{i}}$. There are two further cases:

3.b.ii.A. Case $\tilde{\tilde{i}} \in S_c$: Then, $\tilde{\tilde{c}} \neq c$. The instantaneous balance of c will increase by 1, and we will start from the beginning again with $\tilde{\tilde{i}}$ instead of i. The total change in c's balance since the beginning will be 0. Also, no other college's balance has changed since the beginning.

3.b.ii.B. Case $\tilde{\tilde{i}} \notin S_c$: We start from Step 3.b above with student $\tilde{\tilde{i}}$ instead of $\tilde{i}$.

Thus, whenever we continue from the beginning, the instantaneous balance of c is $b_c^{\hat{\pi}}$, and whenever we continue from Step 3.b, the instantaneous balance of c is $b_c^{\hat{\pi}} - 1$ or $b_c^{\hat{\pi}}$ and the instantaneous balances of all other colleges either increase by one or stay the same. Due to finiteness, the algorithm will terminate at some point at Steps 1 or 2 or 3.b.i; and the net balance of c at the new DA outcome will be $b_c^{\hat{\pi}}$ or $b_c^{\hat{\pi}} - 1$. Moreover, whenever the algorithm terminates, the net balance of any other college has gone up by one or stayed the same. ■

We are ready to prove the results stated in the Appendix A.

Proof of Proposition 6. First recall that, any stable matching for the associated college admission market of a tuition exchange market is also a stable matching for that tuition exchange market. Let $\left[\hat{q}, P_S, \overline{P}_C\right]$ and $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}'_C\right]$ be the associated college admissions markets of $\left[\hat{q}, \hat{e}, \succsim\right]$ and $\left[(\tilde{q}_c, \hat{q}_{-c}), (\hat{e}_c + 1, \hat{e}_{-c}), \succsim\right]$, respectively. Let $\hat{\pi}$ and $\tilde{\pi}$ be the outcome of DA for $\left[\hat{q}, P_S, \overline{P}_C\right]$ and $\left[(\tilde{q}_c, \hat{q}_{-c}), P_S, \overline{P}'_C\right]$, respectively. By Propositions 3 and 5, it is sufficient to prove the proposition for $\hat{\pi}$ and $\tilde{\pi}$. Note that $M_c^{\hat{\pi}} = \hat{\pi}(c)$ by Assumption 3 Part 3, and $\hat{\pi}$ is stable in $\left[\hat{q}, \hat{e}, \succsim\right]$.

Two cases are possible:

Case 1: $b_c^{\hat{\pi}} < 0$: We have $|\hat{\pi}(c)| = |M_c^{\hat{\pi}}| < |X_c^{\hat{\pi}}| \leq \hat{e}_c \leq \hat{q}_c$. Then, by Lemma 2, $b_c^{\tilde{\pi}} \in \{b_c^{\hat{\pi}} - 1, b_c^{\hat{\pi}}\}$.

Case 2: $b_c^{\hat{\pi}} \geq 0$: We have two cases again:

2.a. $|\hat{\pi}(c)| < \hat{q}_c$ or $\tilde{q}_c = \hat{q}_c$: By Lemma 2, $b_c^{\tilde{\pi}} \in \{b_c^{\hat{\pi}} - 1, b_c^{\hat{\pi}}\}$.

2.b. $|\hat{\pi}(c)| = \hat{q}_c$ and $\tilde{q}_c = \hat{q}_c + k$ for $k > 0$: Denote the newly certified student of c by i in market $\left[(\tilde{q}_c, \hat{q}_{-c}), (\tilde{e}_c, \hat{e}_{-c}), \succsim\right]$. We first consider the outcome of DA in the associated college admissions market of $\left[(\tilde{q}_c, \hat{q}_{-c}), \hat{e}, \succsim\right]$, which we denote by π''. We first show that the number of students imported by c in π'' cannot be less than the one in $\hat{\pi}$. Let $C_< = \{\tilde{c} \in C : |\hat{\pi}(\tilde{c})| < \hat{q}_{\tilde{c}}\}$. By our construction, in any stable matching for the associated college admissions market all students in $S \setminus E$ are assigned to $c_\emptyset$ where E is the set of eligible students according to $\hat{e}$. Due to the nonwastefulness of $\hat{\pi}$, $\hat{\pi}(s) P_s \tilde{c}$

for all $s \in E \setminus \hat{\pi}(\tilde{c})$ and $\tilde{c} \in C_<$. We know that DA is resource monotonic: when the number of seats (weakly) increases at each college, then every student will be weakly better off (see Kesten, 2006). That is, $\pi''(s) R_s \hat{\pi}(s)$ for all $s \in E$. By combining the resource monotonicity and individual rationality of DA, we can say that if a student is assigned to a college in $\hat{\pi}$, then she will also be assigned to a college in π''. Hence, we can write:

$$\sum_{c' \in C} |\pi''(c')| \geq \sum_{c' \in C} |\hat{\pi}(c')|. \tag{1}$$

Note that the difference between the left-hand side and the right-hand side of the equation can be at most k. This follows from the fact that in π'' no new student will be assigned to a college in $C_<$, the number of students assigned to other colleges can increase only for c, and the maximum increment is k.

By combining nonwastefulness and resource monotonicity we can write:

$$\sum_{\tilde{c} \in C_<} |\pi''(\tilde{c})| \leq \sum_{\tilde{c} \in C_<} |\hat{\pi}(\tilde{c})|. \tag{2}$$

Then, if we subtract the left-hand side of Equation 2 from the left-hand side of Equation 1 and the right-hand side of Equation 2 from the right-hand side of Equation 1, we get the following inequality:

$$\sum_{c' \in C \setminus C_<} |\pi''(c')| \geq \sum_{c' \in C \setminus C_<} |\hat{\pi}(c')|. \tag{3}$$

Given that each college in $C \setminus C_<$ fills its seats in $\hat{\pi}$, when we subtract $\sum\limits_{c' \in C \setminus (C_< \cup c)} \hat{q}_{c'}$ from both sides of Equation 3, we get the following inequality:

$$|\pi''(c)| + \sum_{c' \in C \setminus (C_< \cup c)} (|\pi''(c')| - \hat{q}_{c'}) \geq |\hat{\pi}(c)|. \tag{4}$$

The term $\sum\limits_{c' \in C \setminus (C_< \cup c)} (|\pi''(c')| - \hat{q}_{c'})$ is nonpositive since $|\pi''(c')| \leq \hat{q}_{c'}$ for all $c' \in C \setminus (C_< \cup c)$. Therefore, $|\pi''(c)| \geq |\hat{\pi}(c)|$.

If $|\pi''(c)| = |\hat{\pi}(c)|$ then $|\pi''(c')| = |\hat{\pi}(c')|$ for all $c' \in C$. This follows from Equation 4, Equation 2, and the fact that $|\pi''(c')| \leq |\hat{\pi}(c')|$ for all $c' \in C \setminus \{c\}$. Therefore, c cannot export and import more students, and $b_c^{\pi''} = b_c^{\hat{\pi}}$. If $|\pi''(c)| > |\hat{\pi}(c)|$, then at most k more students can be assigned to a college in π'' among the eligible students who were not assigned to a college in $\hat{\pi}$. It is possible that some of the students belong to S_c. Thus, $b_c^{\pi''} \in \{b_c^{\hat{\pi}}, ..., b_c^{\hat{\pi}} + k\}$.

By Lemma 2, as we increase the eligibility quota of college c by 1 and keep the admission quota at $\tilde{q}_c$, we have $b_c^{\tilde{\pi}} \in \{b_c^{\pi''} - 1, b_c^{\pi''}\}$, and hence, $b_c^{\tilde{\pi}} \in \{b_c^{\hat{\pi}} - 1, b_c^{\hat{\pi}}, ..., b_c^{\hat{\pi}} + k\}$. ■

Proof of Theorem 10. Given Proposition 6, when c decreases its certification quota by one and keeps its admission quota the same, its balance in any stable matching for the new market will either be the same or increase by one. Since c will have a nonnegative balance in any stable matching for the market $\left[\hat{q}, (\bar{e}_c = 0, \hat{e}_{-c}), \succsim\right]$, there exists $0 \leq \tilde{e}_c \leq \hat{e}_c$ such that c has a zero-balance in every stable matching for the market $\left[\hat{q}, (\tilde{e}_c, \hat{e}_{-c}), \succsim\right]$. ■

Proof of Theorem 11. We consider two markets: $\left[\hat{q}, \hat{e}, \succsim\right]$ and $\left[(q'_c, \hat{q}_{-c}), (\hat{e}_c - 1, \hat{e}_{-c}), \succsim\right]$ with $\hat{q}_c \geq \hat{e}_c$ and $\hat{q}_c \geq q'_c \geq \hat{e}_c - 1$ such that for c, $b_c^{\mu} < 0$ for a stable matching μ for the first market. Let μ' be an arbitrary stable matching for the second market. We want to show that $b_{-c}^{\mu} \geq b_{-c}^{\mu'}$. From Proposition 6, we know that $b_c^{\mu'} < 0$ or $b_c^{\mu'} = 0$. By Proposition 5, without loss of generality we assume that μ and μ' are the outcome of the sequential DA algorithm for the associated college admissions market of $\left[\hat{q}, \hat{e}, \succsim\right]$ and $\left[(q'_c, \hat{q}_{-c}), (\hat{e}_c - 1, \hat{e}_{-c}), \succsim\right]$, respectively. We have two cases:

Case 1: $b_c^{\mu'} < 0$. We have $|\mu'(c)| = |M_c^{\mu'}| < |X_c^{\mu'}| \leq \hat{e}_c - 1 \leq \min\{\hat{q}_c, q'_c\}$. Hence, as c did not fill its admission quota at μ' under both $\hat{q}_c$ and q'_c, in market $\left[\hat{q}, (\hat{e}_c - 1, \hat{e}_{-c}), \succsim\right]$ μ' will still be the outcome of DA for the associated college admissions market. When we add a new student i from c to the set of eligible students, we obtain $\left[\hat{q}, \hat{e}, \succsim\right]$. By Lemma 2, we have $b_{c'}^{\mu} \in \{b_{c'}^{\mu'}, b_{c'}^{\mu'} + 1\}$ for all $c' \in C \setminus c$.

Case 2: $b_c^{\mu'} = 0$. There are two possibilities: (a) $|\mu'(c)| < q'_c$ and (b) $|\mu'(c)| = q'_c$.

2.a. If $|\mu'(c)| < q'_c$, then by Lemma 2, we have $b_{c'}^{\mu} \in \{b_{c'}^{\mu'}, b_{c'}^{\mu'} + 1\}$ for all $c' \in C \setminus c$.

2.b. If $|\mu'(c)| = q'_c$, then $|\mu'(c)| = \hat{e}_c - 1 = q'_c$.[26] We first increase the admission quota of c from q'_c to $\hat{q}_c$ and keep its eligibility quota at $\hat{e}_c - 1$. Suppose the number of students assigned to c increases at the outcome of DA under the associated college admissions market of $\left[\hat{q}, (\hat{e}_c - 1, \hat{e}_{-c}), \succsim\right]$, which we denote by μ'', i.e., $|\mu''(c)| > |\mu'(c)| = \hat{e}_c - 1$. Thus, $b_c^{\mu''} > 0$. When we also increase the eligibility quota of c from $\hat{e}_c - 1$ to $\hat{e}_c$, then by Lemma 2, $b_c^{\mu} \in \{b_c^{\mu''} - 1, b_c^{\mu''}\}$, and hence, $b_c^{\mu} \geq 0$. However, this contradicts the fact that $b_c^{\mu} < 0$. Therefore, $|\mu''(c)| = |\mu'(c)| = q'_c \leq \hat{q}_c$. Hence, under both associated college admissions markets of $\left[(q'_c, \hat{q}_{-c}), (\hat{e}_c - 1, \hat{e}_{-c}), \succsim\right]$ and $\left[\hat{q}, (\hat{e}_c - 1, \hat{e}_{-c}), \succsim\right]$, DA chooses the same matching, i.e., $\mu'' = \mu'$. When we increase the eligibility quota of c from $\hat{e}_c - 1$ to $\hat{e}_c$ and keep the admission quota at $\hat{q}_c$, DA outcome changes from $\mu'' = \mu'$ to μ for the associated college admissions market. By Lemma 2, we have $b_{c'}^{\mu} \in \{b_{c'}^{\mu'}, b_{c'}^{\mu'} + 1\}$ for all $c' \in C \setminus c$.

In either case, $b_{-c}^{\mu'} \leq b_{-c}^{\mu}$. Moreover, Lemma 2 implies the same conclusion for any

[26] That is, this case is possible when $q'_c = \hat{e}_c - 1$.

market $[(q'_c, \hat{q}_{-c}), (e'_c, \hat{e}_{-c}), \succsim]$ where $e'_c \leq \hat{e}_c - 1$. ■

Appendix F Structure of Stable Matchings

In this Appendix, we inspect the structure of stable matchings. In the college admissions market, there always exist student-optimal and college-optimal Gale-Shapley-stable matchings (see Gale and Shapley, 1962; Roth, 1985).[27] Under Assumption 3, we can guarantee the existence of college- and student-optimal stable tuition-exchange matchings. This result's proof also uses the associated Gale-Shapley college admissions market for each tuition-exchange market and the properties of Gale-Shapley stable matchings in these markets.[28]

Proposition 7 *Under Assumption 3, there exist college- and student-optimal matchings in any tuition-exchange market.*

Proof of Proposition 7. By the proof of Proposition 3, Gale and Shapley (1962), and Roth (1985), there exists a student-optimal stable matching for each tuition-exchange market. By Assumption 3 Part 1 and Proposition 5, colleges compare only the stable matchings through the admitted set of students. By Gale and Shapley (1962) and Roth (1985), there exists a college-optimal stable matching for each tuition-exchange market. ■

Appendix G Further Discussion on 2S-TTC Mechanism

We illustrate the dynamics of the 2S-TTC mechanism with an example below.

Example 1 Let $C = \{a, b, c, d, e\}$, $S_a = \{\mathbf{1}, \mathbf{2}\}$, $S_b = \{\mathbf{3}, \mathbf{4}\}$, $S_c = \{\mathbf{5}, \mathbf{6}\}$, $S_d = \{\mathbf{7}, \mathbf{8}\}$, and $S_e = \{\mathbf{9}\}$. Let $e = (2, 2, 2, 2, 1)$ and $q = (2, 2, 2, 1, 1)$. The internal priorities and the

[27] A matching is student-(or college-)optimal stable if it is preferred to all the other stable matchings by all students (or colleges).

[28] The lattice property of Gale-Shapley-stable college-admissions matchings can also be used to prove an analogous lattice property for stable matchings in tuition-exchange markets under Assumption 3. We skip it for brevity.

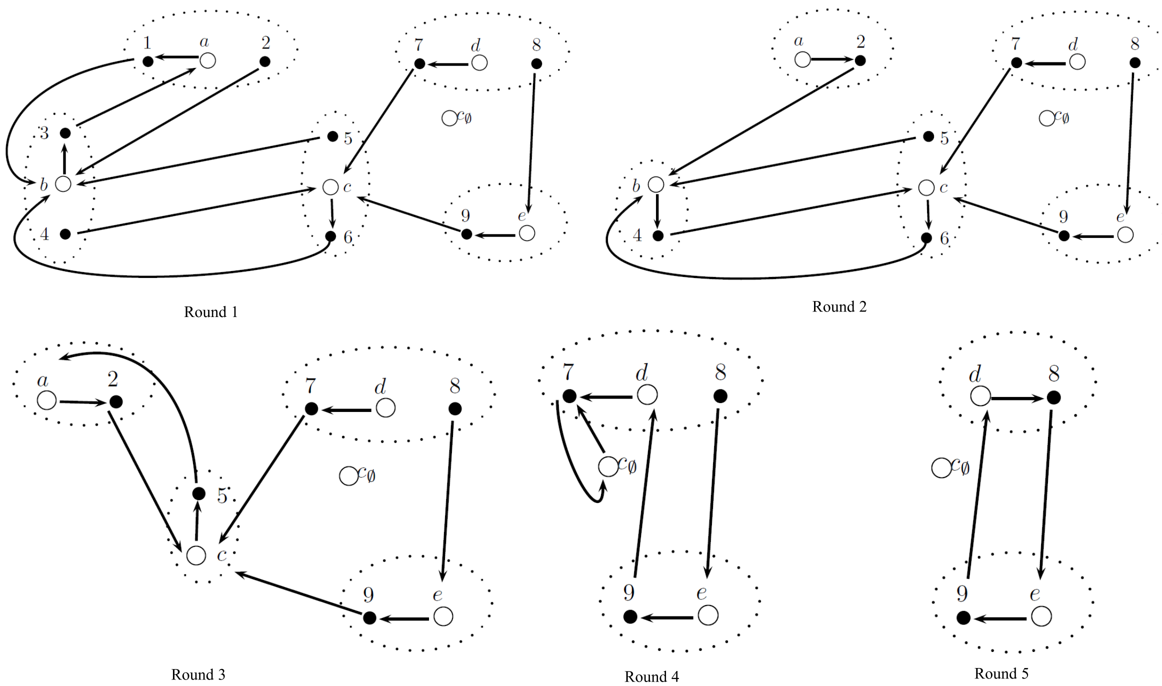
rankings of agents associated with their preferences over matchings are given as:

$\rhd_a$	$\rhd_b$	$\rhd_c$	$\rhd_d$	$\rhd_e$
1	**3**	**6**	**7**	**9**
2	**4**	**5**	**8**	

P_a	P_b	P_c	P_d	P_e
3	**5**	**2**	**2**	**2**
4	**1**	**3**	**3**	**3**
5	**6**	**4**	**4**	**8**
9	**2**	**9**	**9**	**7**
7	**7**	**7**	**5**	**5**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$P_{\mathbf{1}}$	$P_{\mathbf{2}}$	$P_{\mathbf{3}}$	$P_{\mathbf{4}}$	$P_{\mathbf{5}}$	$P_{\mathbf{6}}$	$P_{\mathbf{7}}$	$P_{\mathbf{8}}$	$P_{\mathbf{9}}$
b	b	a	c	b	a	c	e	c
c	c	c	a	a	b	a	c	d
$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$

Let o_e and o_a be the vectors representing the eligibility and admission counters of colleges, respectively. Then we set $o_e = (2, 2, 2, 2, 1)$ and $o_a = (2, 2, 2, 1, 1)$.



Round 1: The only cycle formed is $(b, \mathbf{3}, a, \mathbf{1})$. Therefore, **1** is assigned to b and **3** is assigned to a. Observe that although college a is the most-preferred college of student **6**, she is not acceptable to a, and hence, she points to college b instead. The updated counters are $o_e = (1, 1, 2, 2, 1)$ and $o_a = (1, 1, 2, 1, 1)$.

Round 2: The only cycle formed in Round 2 is $(c, \mathbf{6}, b, \mathbf{4})$. Therefore, **6** is assigned to b and **4** is assigned to c. The updated counters are $o_e = (1, 0, 1, 2, 1)$ and $o_a = (1, 0, 1, 1, 1)$. College b is removed.

Round 3: The only cycle formed in Round 3 is $(a, \mathbf{2}, c, \mathbf{5})$. Therefore, **5** is assigned to a and **2** is assigned to c. The updated counters are $o_e = (0, 0, 0, 2, 1)$ and $o_a = (0, 0, 0, 1, 1)$.

Colleges a and c are removed.

Round 4: The only cycle formed in Round 4 is $(c_\emptyset, \mathbf{7})$. Therefore, $\mathbf{7}$ is assigned to $c_\emptyset$. Given that we have a trivial cycle including $c_\emptyset$, we only update o_e. The updated counters are $o_e = (0,0,0,1,1)$ and $o_a = (0,0,0,1,1)$.

Round 5: The only cycle formed at this round is $(e, \mathbf{9}, d, \mathbf{8})$. Therefore, $\mathbf{8}$ is assigned to e and $\mathbf{9}$ is assigned to d. The updated counters are $o_e = (0,0,0,0,0)$ and $o_a = (0,0,0,0,0)$.

All students are assigned, so the algorithm terminates and its outcome is given by matching

$$\mu = \left(\begin{array}{ccccc} a & b & c & d & e \\ \{\mathbf{3},\mathbf{5}\} & \{\mathbf{1},\mathbf{6}\} & \{\mathbf{2},\mathbf{4}\} & \mathbf{9} & \mathbf{8} \end{array} \right).$$

◇

It would be good to point out a few simple observations regarding regular TTC and 2S-TTC. Since students may not be able to point to their top available choices during the algorithm (as such colleges may find them unacceptable), 2S-TTC is not balanced-efficient for students in general. Since colleges cannot necessarily choose among their acceptable choices, 2S-TTC is not balanced-efficient for colleges in general, either.[29] As this is a two-sided matching market, we could also propose the *college-pointing* version of the 2S-TTC mechanism in which colleges point to their highest ranked students under P_C the ones considering them acceptable and each student points to her home college in each round. This variant takes college preference intensity more seriously. However, it gives incentives to both students and colleges for manipulation. On the other hand, 2S-TTC is group strategy-proof for students, as we state in Theorem 3.

On the other hand, regular TTC mechanism that ignores colleges' preferences all together is balanced-efficient for students (observe that during the TTC algorithm, students always point to their top available college). Its Pareto efficiency in a one-sided market directly implies this result, while this does not provide any immediate efficiency implication for 2S-TTC (other than that among *acceptable* balanced matchings, its outcome is Pareto undominated). Hence, regular TTC is also balanced-efficient for all agents since student preferences are strict. But in general, regular TTC is not acceptable unlike 2S-TTC, as college preferences are ignored altogether. Thus, regular TTC is not a good mechanism for our purposes. We illustrate with an example that 2S-TTC is not balanced-efficient for students and not balanced-efficient for colleges. However, it is balanced-efficient overall

[29] This is in vein similar to the well-known fact that a stable matching is neither efficient for students nor efficient for colleges, in general. But under strict preferences, all stable matchings are Pareto efficient for all agents.

as proven in Theorem 1.

Example 2 *Suppose $C = \{a, b, c\}$ such that $S_a = \{\mathbf{1}\}$, $S_b = \{\mathbf{2}\}$, and $S_c = \{\mathbf{3}\}$. Let $e = q = (1, 1, 1)$. The preferences of students and colleges ranking over incoming students are given as follows:*

P_a	P_b	P_c	$P_{\mathbf{1}}$	$P_{\mathbf{2}}$	$P_{\mathbf{3}}$
3	**2**	**1**	b	c	a
2	**3**	**3**	a	a	b
1	$\emptyset$	**2**	c	b	c

If we apply the regular TTC mechanism to this market without taking colleges' preferences into account, the outcome is $\begin{pmatrix} a & b & c \\ \mathbf{3} & \mathbf{1} & \mathbf{2} \end{pmatrix}$. However, this is not acceptable for colleges: college b gets an unacceptable student, $\mathbf{1}$. Our 2S-TTC mechanism does not select this outcome. In fact, its outcome is $\begin{pmatrix} a & b & c \\ \mathbf{1} & \mathbf{3} & \mathbf{2} \end{pmatrix}$. Observe that although this matching is not balanced-efficient for students (the above TTC outcome Pareto dominates it for students) and not balanced-efficient for colleges (since the matching $\begin{pmatrix} a & b & c \\ \mathbf{3} & \mathbf{2} & \mathbf{1} \end{pmatrix}$ Pareto dominates it for colleges), it is balanced-efficient for all agents. $\diamond$

Appendix H Independence of Axioms

- *A student-strategy-proof, acceptable but not balanced-efficient mechanism that also respects internal priorities:* A mechanism that always selects the null matching for any market.[30]

- *A student-strategy-proof, balanced-efficient, acceptable mechanism that does not respect internal priorities:* Consider a variant of the 2S-TTC mechanism in which each college points to the certified student who has the lowest priority among the certified ones. This mechanism is strategy-proof for students, balanced-efficient, and acceptable, but it fails to respect internal priorities.

- *A balanced-efficient, acceptable, but not student-strategy-proof mechanism that respects internal priorities:* Consider the following market. There are three colleges $C = \{a, b, c\}$ and three students $S_a = \{\mathbf{1}\}$, $S_b = \{\mathbf{2}\}$, and $S_c = \{\mathbf{3}\}$. All students

[30]For the proof of Theorem 9, a mechanism that always assigns workers to their homes firm for any market.

are acceptable for colleges. The ranking P associated with preference profile $\succsim_S$ is given as

$P_{\mathbf{1}}$	$P_{\mathbf{2}}$	$P_{\mathbf{3}}$
b	a	b
c	c	a
a	b	c
$c_\emptyset$	$c_\emptyset$	$c_\emptyset$

Let mechanism ψ select the same matching as 2S-TTC for each market except the market $\left[q = (1,1,1), e = (1,1,1), \succsim\right]$, and for this market it assigns **1** to c, **2** to a, and **3** to b. This mechanism is balanced-efficient, acceptable, and respecting internal priorities. However, it is not student-strategy-proof, because when **1** reports c unacceptable, ψ will assign **1** to b.

- *A balanced-efficient, student-strategy-proof, but not acceptable mechanism that respects internal priorities:* Consider a variant of 2S-TTC in which students are not restricted to point to those colleges that consider them acceptable. This mechanism is balanced-efficient, student-strategy-proof, and respecting internal priorities, but it is not acceptable since an unacceptable student can be assigned to a college.

Appendix I Simulations

Theoretically, 2S-TTC and the current decentralized market procedure modeled in Appendix A cannot be Pareto ranked. Moreover, when we consider the number of unassigned students, neither 2S-TTC nor the decentralized market procedure performs better than the other in every market. In order to compare the performances of 2S-TTC and the current decentralized market procedure, we run computer simulations under various scenarios. We consider environments with 10 and 20 colleges and 5 and 10 available seats. Each student is linked to a college, and the number of students linked to a college is equal to its capacity. We construct the preference profile of each student $s \in S_c$ by incorporating the possible correlation among students' preferences. In particular, we calculate s's utility from being assigned to college $c' \in C \setminus \{c\}$ as follows:[31]

$$U(s, c') = \beta Z(c') + (1 - \beta) X(s, c').$$

Here, $Z(c') \in (0,1)$ is an i.i.d. standard uniformly distributed random variable and it

[31]To be consistent with the tuition exchange practice, we do not calculate students' utilities for their home colleges.

represents the common tastes of students on c'. $X(s,c') \in (0,1)$ is also an i.i.d. standard uniformly distributed random variable and it represents the individual taste of s on c'. The correlation in the students' preferences is captured by $\beta \in [0,1]$. As β increases, the students' preferences over the colleges become more similar. For each student s we randomly choose a threshold utility value $T(s)$ in order to determine the set of acceptable colleges where $T(s) \in (0, 0.5)$ is an i.i.d. standard uniformly distributed random variable. We say c' is acceptable for s if $T(s) \leq U(s,c')$. By using the utilities students get from each college and the threshold value, we construct the ordinal preferences of students over colleges.

In order to construct college rankings (preferences) over students, we follow a similar method as in the student preference profile construction. In particular, we calculate c's utility from $s' \in S \setminus S_c$ as follows:[32]

$$V(c,s') = \alpha W(s') + (1-\alpha) Y(c,s').$$

Here, $W(s') \in (0,1)$ is an i.i.d. standard uniformly distributed random variable and it represents the common tastes of colleges on s'. $Y(c,s') \in (0,1)$ is also an i.i.d standard uniformly distributed random variable and it represents the individual taste of c on s. The correlation in the college rankings is captured by $\alpha \in [0,1]$. Like β, as α increases the colleges' rankings over the students become more similar. For each $c \in C$ we randomly choose a threshold value $T(c)$ in order to determine the set of acceptable students for c where $T(c) \in (0, 0.5)$ is an i.i.d. standard uniformly distributed random variable. We say s' is acceptable for c if $T(c) \leq V(c,s')$. By using the utilities colleges get from each student and the threshold value, we construct the ordinal rankings of colleges over students.

Under each case, we consider a time horizon of 25 periods.[33] In order to mimic the decentralized procedure, we use student proposing DA mechanism in each period. We consider two different strategies colleges play. Under the first strategy, each college certifies its all students as eligible in period 1. Observe that this is a naive behavior, and in a sense the best-case scenario if colleges are negative-balance averse. Under this assumption, colleges have incentives to certify fewer students than their quota (see Theorems 10 and 11 in Appendix A). For further periods, if a college c carries an aggregate negative balance of x, then it certifies only $q_c - x$ students, otherwise it certifies all its students. Under the second strategy, in each period we rerun the DA mechanism until the outcome in that period satisfies zero balance and in each run a college with negative balance ex-

[32]To be consistent with the tuition exchange practice, we do not calculate colleges' utilities for their own students.

[33]We consider the first five periods as warm-up periods.

cludes one student from its certified list. On the other hand, under 2S-TTC, since colleges run a zero balance, each college certifies all of its students in each period. Under each scenario, we run the DA and TTC 1,000 times by using different random draws for X, Y, W, Z, and T and calculate the number of students unassigned under DA and 2S-TTC, and the number of students preferring 2S-TTC over DA and vice versa. For each run, we use the same draw of Z for all 25 periods.

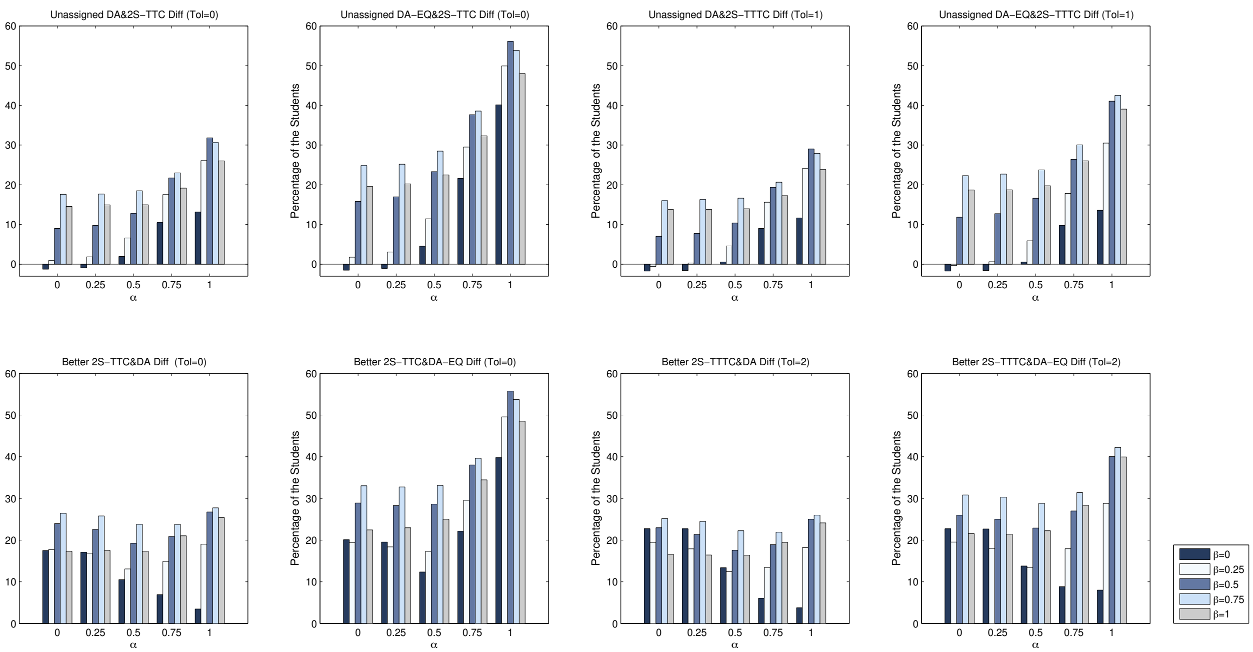


Figure 1: Student welfare under simulations with 20 colleges each with 10 seats

We also relax the zero-balance constraint and allow each college to run a negative balance of not more than 20% of its quota. Under the first strategy for a decentralized market, each college certifies all its students as eligible in period 1. For further periods, if a college carries an aggregate negative balance of $x > 0.2q_c$, then it certifies only $1.2q_c - x$ students, otherwise it certifies all its students. Under the second strategy, we exclude students from each college's certification set only if they run a negative balance more than 20% of their quota. Similarly, since each college can run a certain amount of negative balance, we use 2S-TTTC instead of 2S-TTC with a tolerable balance interval of $[-0.2q_c, \infty)$.

In Figure 1, we illustrate the simulation results for 20 colleges and 10 seats case. The horizontal axis refers to changing levels of α and β, the preference correlation parameters. Different values of β are grouped together (shown in the right-bottom graph's legend) while α is used as the main horizontal axis variable. The vertical axis variables in top 4 graphs demonstrate the difference of the percentage of unassigned students between the DA mechanism under the two alternative strategies of the colleges (In each row, the 1st and 3rd graphs are for straightforward behavior of DA, i.e., strategy 1, and the 2nd and 4th graphs are for the equilibrium behavior of DA, i.e., strategy 2, explained above) and 2S-TTC/2S-TTTC. In bottom 4 graphs, the vertical axes demonstrate the difference between the percentage of the students preferring the versions of 2S-TTC and the percentage of the students preferring the DA mechanism under two alternative strategies of the colleges.[34]

Under all scenarios, when we compare the percentage of students preferring the versions of 2S-TTC and the DA mechanism under two alternative strategies of the colleges, we observe that 2S-TTC and 2S-TTTC outperform both alternative strategic behaviors under DA. For example, when $\alpha = 0.5$ and $\beta = 0.5$, for yearly tolerance level 0, 19.23% more of all students (i.e., the percentage of students who prefer 2S-TTC to DA minus the percentage who prefer DA to 2S-TTC) prefer 2S-TTC outcome to DA straightforward behavior outcome (while this difference increases to 28.62% for DA equilibrium simulations), as seen in the middle of the graph of the 1st (and 2nd, respectively) graph of the bottom row of Figure 1.[35]

Except for very low correlation in both college and student preferences, we observe that the percentage of unassigned students is less under the versions of 2S-TTC compared to the one under both alternative strategic behaviors under DA. For example, when $\alpha = 0.5$ and $\beta = 0.5$, for the yearly tolerance level 0, 2S-TTC matches 12.74% of all students more over the percentage matched by DA under straightforward behavior (while this difference increases to 23.30% over the percentage matched by DA under equilibrium behavior) as seen in the middle of the 1st graph (and 2nd graph, respectively) in the top row of Figure 1, respectively.[36]

In general, as α, the colleges' preference correlation parameter, increases, both welfare measures favor 2S-TTC over DA increasingly more under both tolerance level and both

[34] The results of the other cases are illustrated in Figures 4-6.

[35] We do not give a separate figure for the absolute percentage of students who prefer 2S-TTC over DA. For the considered tolerance 0 scenarios of Figure 1, the absolute percentage of students who prefer 2S-TTC over DA-straightforward treatment changes between 20% to 49.5% for different levels of α and β – minimized at $\alpha = 1$ and $\beta = 0$, while maximized at $\alpha = 0$ and $\beta = 0.75$ (and $23\% - 58.5\%$ of students prefer 2S-TTC to DA-equilibrium treatment – minimized at $\alpha = 0.5$ and $\beta = 0$, while maximized at $\alpha = 1$ and $\beta = 0.5$).

[36] Although, we do not give a separate figure, it is noteworthy to mention that the absolute percentage of students unmatched under DA straightforward scenario increases from 1.4% to 58.4% of all students in both α and β per period, under tolerance 0 scenarios of Figure 1. The corresponding percentages are 1.1% to 80.5% under the DA equilibrium scenario, increasing again in both α and β. Other treatments, including the ones reported at the end of this section, display similar pattern although percentage change interval is slightly different.

DA behavior scenarios. On the other hand, as β, the students' preference correlation parameter, increases, 2S-TTC's dominance measures display mostly a unimodal pattern (peaking for moderate β) for any fixed α. We conclude that 2S-TTC and 2S-TTTC approaches outperform DA methods in almost all cases.

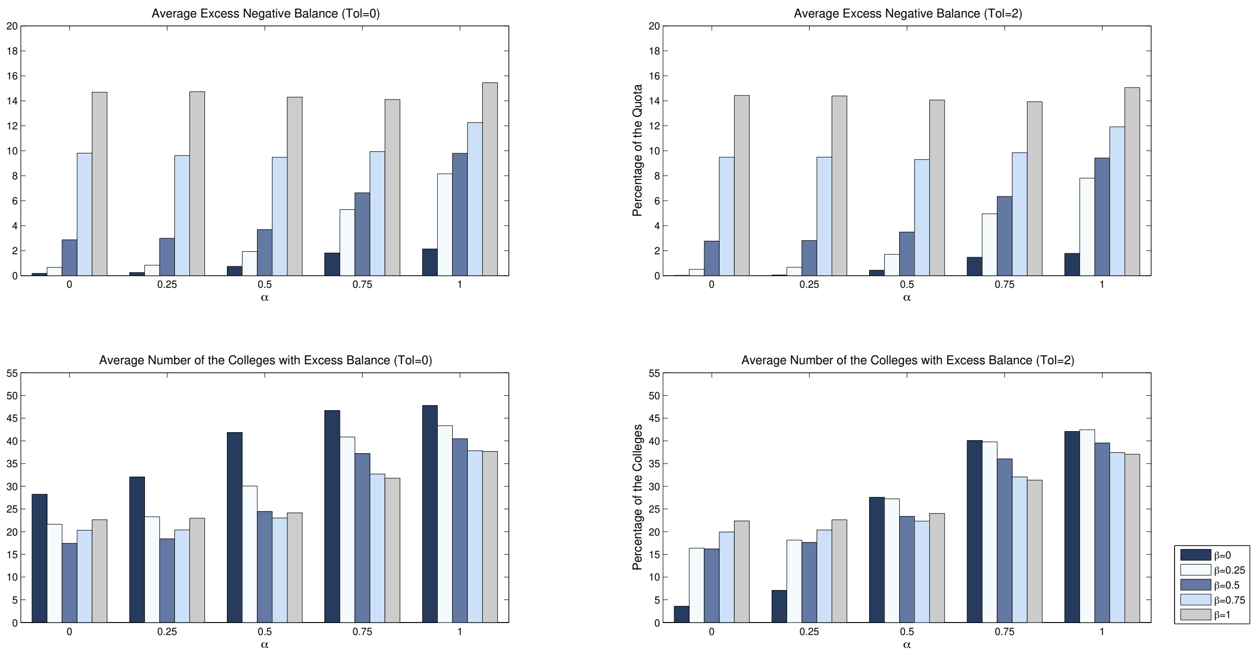


Figure 2: Excess balance under DA when all students are eligible; simulation results for markets with 20 colleges each with 10 seats

One may think that when colleges do not limit the number of eligible students, they would achieve tolerable balance levels eventually. To test this claim, we run DA mechanism when colleges do not limit the number of their eligible students. We calculate (1) the percentage of colleges with excess negative balance at the end of the time horizon and (2) the magnitude of the total excess negative balance relative to the total number of available seats in all periods. The case for 20 colleges and 10 seats is given in Figure 2. The average negative balance of colleges varies between 0.2% and 15% of the available seats at colleges and increases with α and β. Similarly, as α increases the percentage of colleges with excess negative balance increases and it varies between 17% and 45%.[37]

Finally, we consider the case in which the number of students applying to be certified by each college varies in each period. In particular, we run simulations for 10 colleges

[37]The results of all other cases are illustrated in Figures 7-9.

and each college has 10 available seats. Different from the previous cases, the number of students applying to be certified may vary and it is selected from interval $[6, 10]$ according to i.i.d. uniform distribution. Preference profiles of the students and the colleges are constructed as described above. We measure the performances of 2S-TTC and 2S-TTTC compared to the DA mechanism under two strategic behaviors. The results are illustrated in Figure 3. All the results are consistent with the cases in which the number of students in each college equals to the number of available seats.

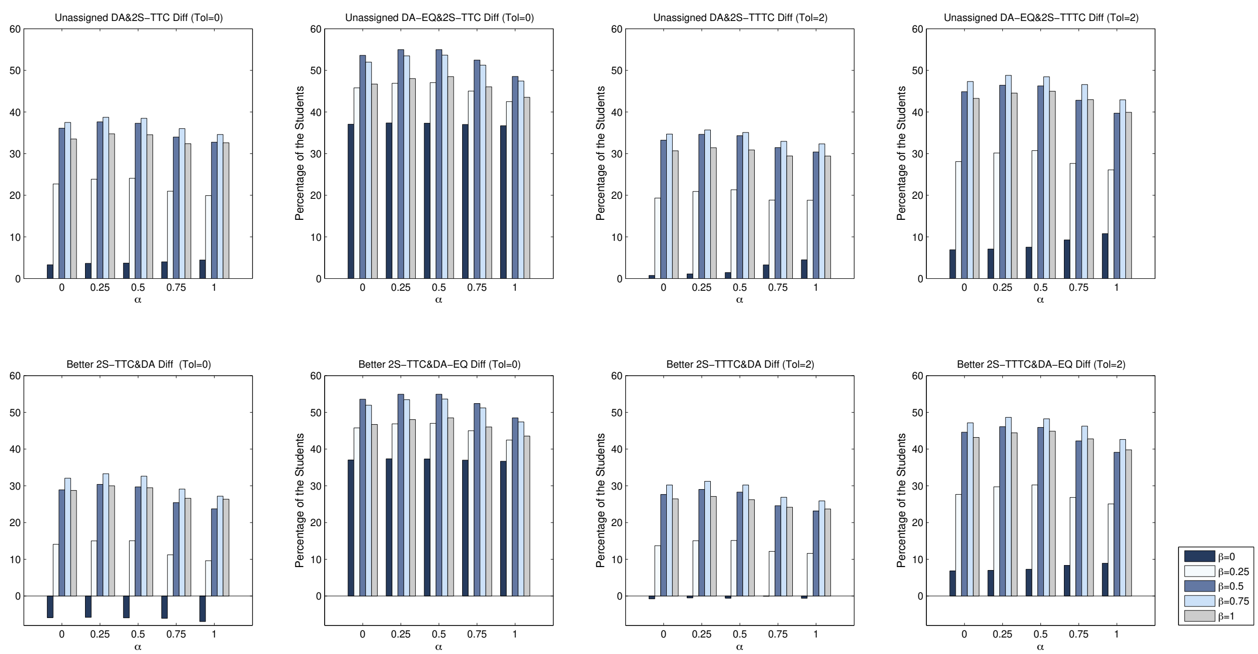


Figure 3: Student welfare in unbalanced market under simulations with 10 colleges each with 10 seats.

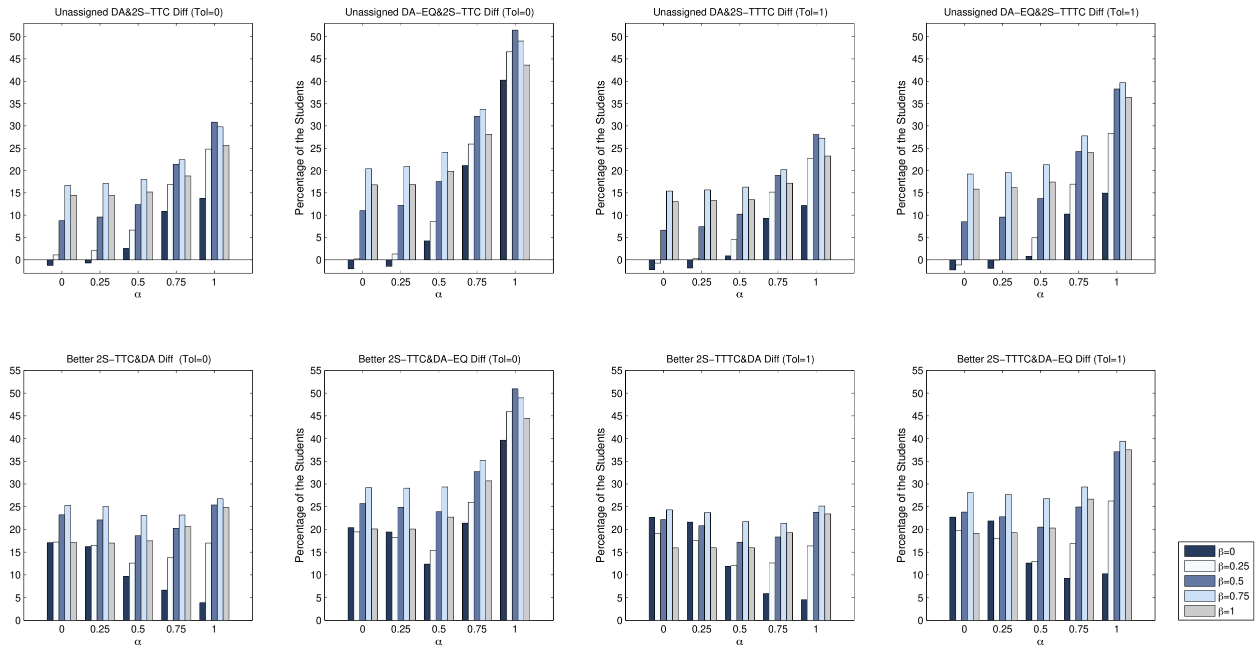


Figure 4: Student welfare under simulations with 20 colleges each with 5 seats.

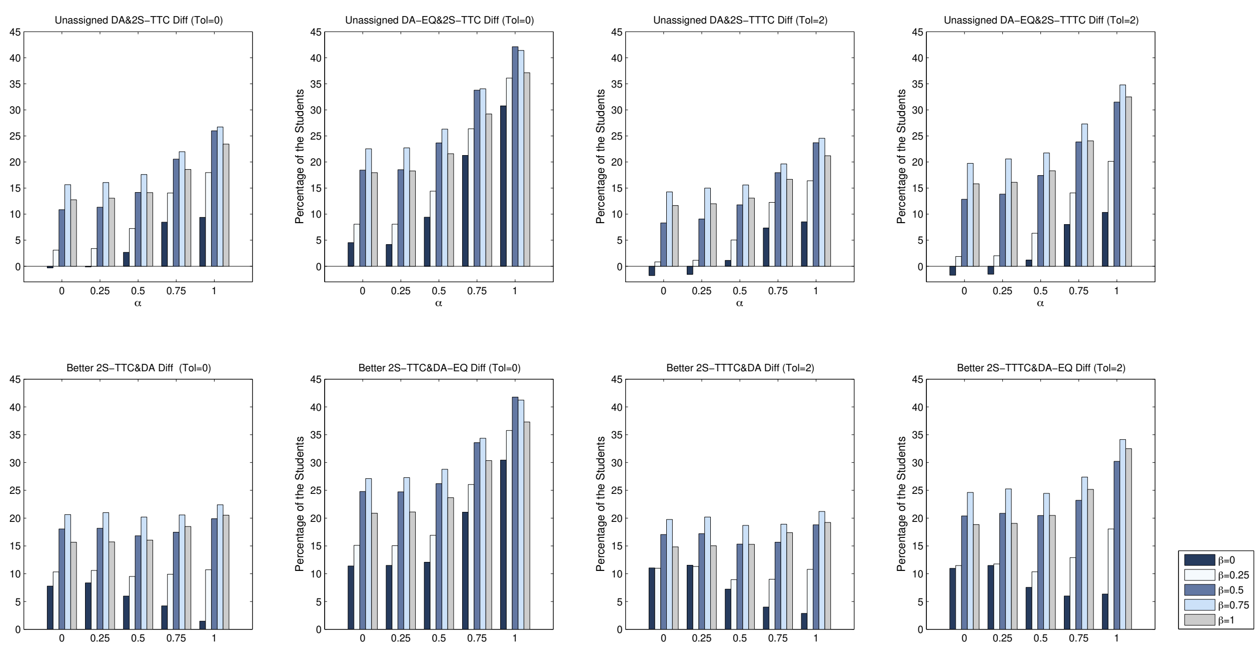


Figure 5: Student welfare under simulations with 10 colleges each with 10 seats.

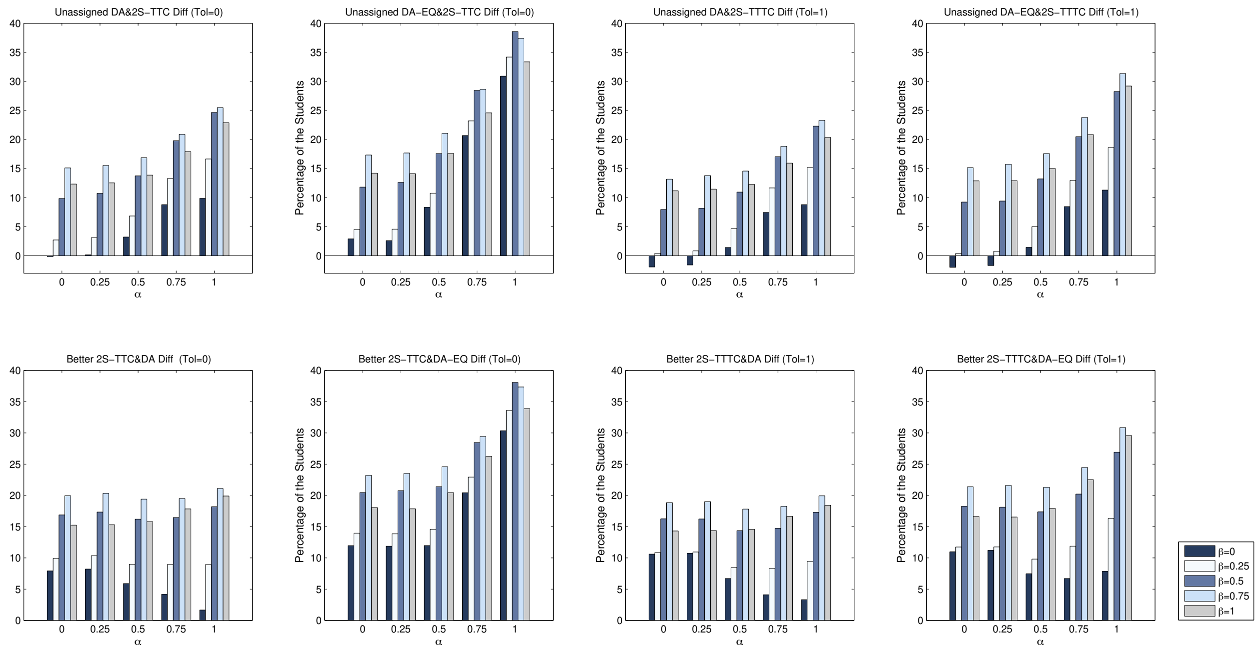


Figure 6: Student welfare under simulations with 10 colleges each with 5 seats.

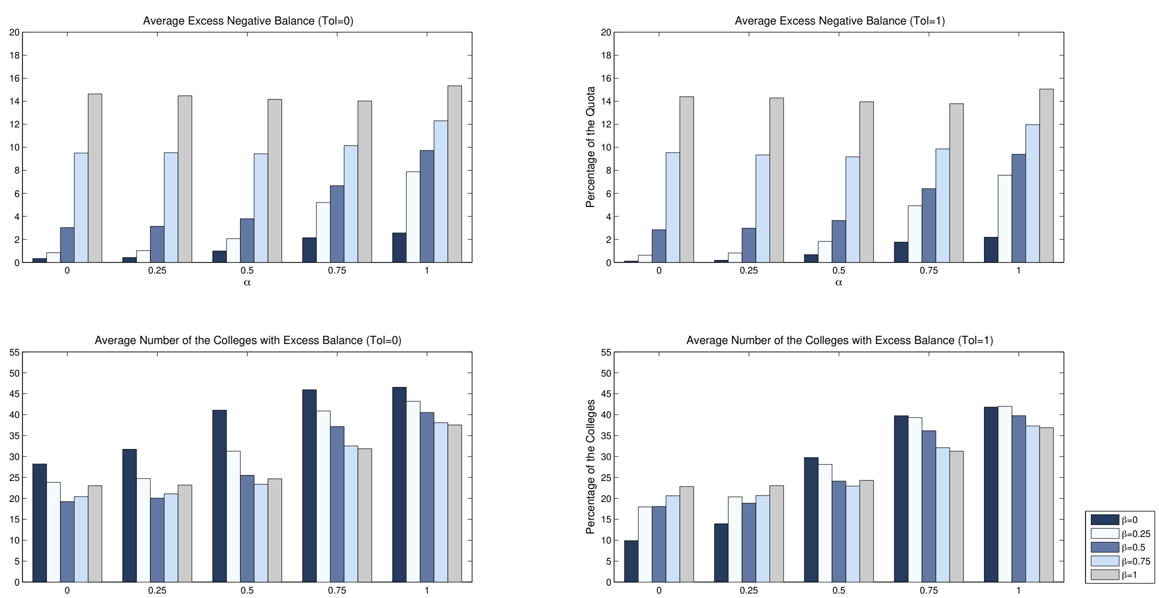


Figure 7: Excess balance under DA when all students are eligible; simulations with 20 colleges each with 5 seats.

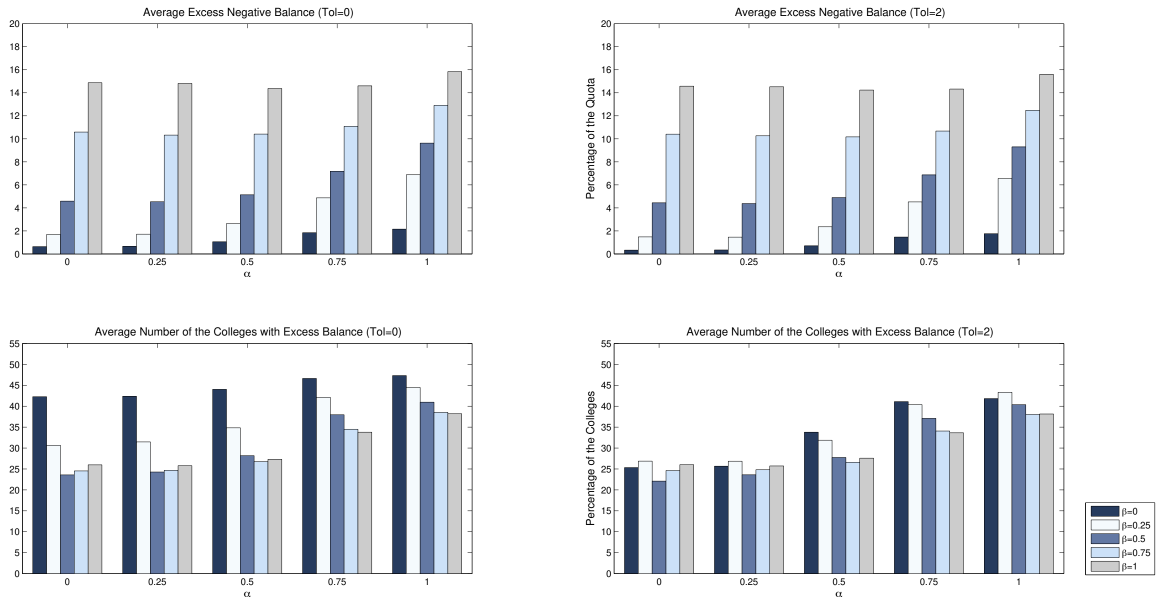


Figure 8: Excess balance under DA when all students are eligible; simulations with 10 colleges each with 10 seats.

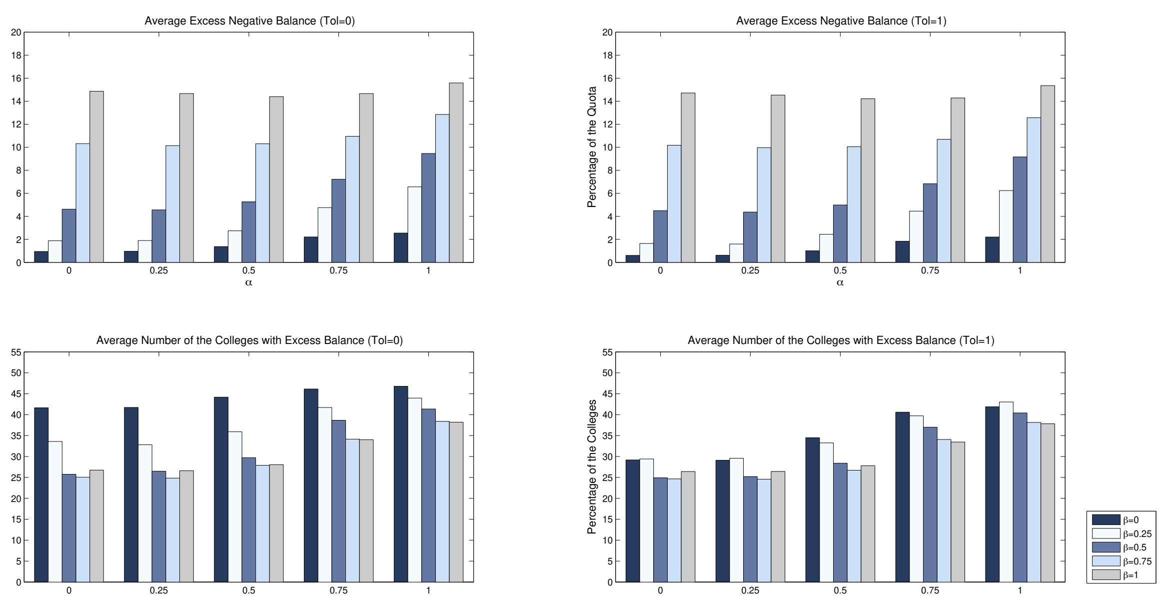


Figure 9: Excess balance under DA when all students are eligible; simulations with 10 colleges each with 5 seats.

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