Allocation Rules of Indivisible Prizes in Team Contests*

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Abstract

We analyze contests in which teams compete to win indivisible homogeneous prizes. Teams are composed of members who may differ in their ability, and who exert effort to increase the success of their team. Each team member can obtain at most one prize as a reward. As effort is costly, teams use the allocation of prizes to give incentives and solve the free-riding problem. We develop a two-stage game. First, teams select a prize-allocation rule. Then, team members exert effort. Members take into account how their effort and the allocation rule influence the chance they receive a prize. We prove the existence and uniqueness of equilibrium. We characterize the optimal prize-assignment rule and individual and aggregate efforts. We then show that the optimal assignment rule is generally not monotonic.

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1 Introduction

Several teams are competing to win one or more prizes. Members of a team exert individual unobservable costly effort. Aggregate team effort influences the number of prizes won by the team. Free-riding incentives lead to the standard moral-hazard in teams problem (see Holmstrom 1982). The way the team allocates prizes won to its members generates incentives and is crucial to increase team effort and team success.

We analyze the optimal **intra-team prize allocation rule** when teams compete for indivisible homogeneous prizes, individual effort is not observable, all prizes won must be allocated between team members, and each team member can obtain at most one prize. Compared to standard models of moral hazard in teams, we do not allow for monetary transfers and all incentives are generated by the prospect of winning a prize.

The main application of the model is electoral competition under (closed-list) proportional representation. In this electoral system, citizens do not cast votes for individual candidates but for a party and its list of candidates. Candidates exert effort to increase the electoral appeal of their party. Prizes are parliament seats. Parties win a random number of seats depending on the aggregate effort of their candidates. Candidates, who may differ in their ability and in their cost of effort, are motivated by their desire to enter parliament. Since parliament seats are homogeneous indivisible prizes, an allocation rule selects which candidates go to parliament for each realized number of seats the party wins. One natural assignment rule is the (closed) *list rule*. A list rule orders candidates, and when n seats are won by the party, the first n candidates on the list go to parliament. Closed lists are used in many democracies relying on proportional representation. In this paper, we do not restrict attention to list rules and derive the optimal assignment rule that maximizes the team's aggregate effort and expected number of prizes won. We allow for any rule that maps the number of prizes won into a subset of team members with the same cardinality. We also derive conditions under which the optimal rule is a list rule.

The interplay between intra-team incentives and inter-team competition is also relevant in other contexts. For instance, consider two departments within a firm. These departments can be viewed as teams.

The firm's CEO designs the hiring and promotion policy to provide incentives to their various employees. As performance typically correlates positively with the need to expand the size of a department, the number of positions and promotions going to each department are proportional to the department's relative performance. Then, employees in a department exert effort to boost their department performance. Each of them hopes to receive one of the prizes won by their department. In this context, what intradepartmental prize allocation rule should the firm use to maximize team effort?

Another example to consider is the situation in which a central government needs to choose where to build several local public goods, such as hospitals, schools and/or military bases. As the local public goods generate both direct and indirect jobs and economic activity for the region in which they are located, each region in the country would like to see the public goods built on their territory. The different regions thus lobby to secure the local public goods. Yet, once a region has won some of these goods, the exact location of these are to be decided among the region's different cities or municipalities. Regions can thus be described as teams and, once again, to pin down which intra-team allocation rule maximizes team output, we need to study the interplay between incentives within regions as well as competition between regions.

We model the contest as a two-stage game. In stage 1, each team chooses simultaneously and independently a prize assignment rule to maximize the expected number of prizes it wins. In stage 2, all team members exert effort to maximize their expected payoff. We assume that the aggregate effort of a team is the sum of their members' individual efforts. Team members differ in the productivity of their effort and the cost of exerting effort. Propositions 1 to 3 characterize the equilibrium of the second stage. We characterize equilibrium efforts for given probabilities of winning for the teams. Then we show through a fixed point argument that there exists a unique Nash equilibrium for given allocation rules. We then move to the choice of the prize allocation rule, in stage 1. Theorem 1 provides sufficient conditions for the existence and uniqueness of equilibrium. Theorem 2 characterizes optimal prize-assignment rules. We need to consider two cases separately, depending on the convexity of the cost function. When the cost function is less convex than a quadratic function, the optimal allocation rule gives the highest incentives to exert effort to the best team members (in terms of productivity over cost). Theorem 2 provides an algorithm to compute the optimal allocation rule. When the cost function is more convex, equalizing incentives becomes more important and the characterization is less clear-cut. In both cases, the optimal allocation rule is usually non-monotonic in the sense that the probability that a given team member wins a prize is not always increasing in the number of prizes won by their team. When we impose monotonicity, list rules become optimal and we characterize the optimal list. We show that in general the best candidates are not put at the top of the list.

The rest of the paper is organized as follows. Section 2 presents a brief review of the literature. Section 3 presents the model. Section 4 solves for the equilibrium choice of effort. In section 5, we analyze the choice of allocation rule by teams. We characterize the optimal rules. subgame perfect equilibria of this game. Section 6 discusses the implications of our results.

2 A Brief Literature Review

The paper contributes to the literature on moral hazard and free-riding in teams. (See Alchian and Demsetz, 1972, Holmstrom 1982, and Olson 1971). More specifically, our paper belongs to the literature on team contests and creates a bridge between two different strands of the literature on contests[1]: team contests and contests for multiple prizes. In team contests, several teams compete in order to win one prize, which may be of a public or private nature, or a mix of both. This strand of the literature focuses on incentives within teams and more specifically on the sharing rule that splits the single available (private part of the) prize across the winning team's members, so as to maximize team output. Important contributions include Nitzan (1991), Esteban and Ray (2001), Nitzan and Ueda (2011), Balart et al. (2016) and Trevisan (2020). In the literature on team contests, there are a few papers that analyzed the winning-probability-maximizing prize-allocation rule when a team effort aggregator function is a CES or linear function while team members have strictly convex effort cost functions with the same effort elasticity of cost (Crutzen et al. 2020, Simeonov 2020, Kobayashi and Konishi 2021, and Kobayashi et al. 2023), and showed that the optimal prize-allocation rule does not depend on the competitiveness of the contests—it depends only on team's production technologies.

[1]The literature on contests is too vast to be reviewed here. We refer to Corchon (2007), Konrad (2009), Sisak (2009), Flamand and Troumpounis (2015) and Vojnovic (2015) for extensive reviews.

We can also connect the above results with the ones by the predecessors, Nitzan and Ueda (2014) and Esteban and Ray (2001). Another strand of this literature uses the all-pay auction model to analyze competition between teams with incomplete information and public good prize (every team member receives the same reward for the team's winning); see for instance Barbieri and Malueg (2016), Barbieri, et al. (2019), Barbieri and Topolyan (2021), and Eliaz and Wu (2017). In contrast, in our model, players have perfect information about their teammates' abilities, but the rewards are heterogeneous; an allocation rule determines which team members win one of the prizes won by the team. Finally, Fu, Lu, and Pan (2015) analyze a multi-battle team contest in which heterogeneous players from two rival teams form pairwise matches to compete in distinct component battles. Considering the contest from a designer's perpsective, Feng, et al. (2022) study effort-maximizing prize allocation rules by granting a contest organizer full flexibility to reward a team based on the full path of battle outcomes and its identity. They show that the optimal rule takes the form of a majority-score rule with a head start score given to the weaker team. In their model, a winning prize is a public good for the team members, while the prizes are distributed to all participants of the contest as indivisible (identical) private goods in our model.

We also contribute to the literature on contests for multiple prizes. Most of this literature focuses on contests between individuals who can win at most one prize; Clark and Riis (1996) and Barut and Kovenock (1998) analyze the cases with identical and vertically differentiated prizes. The intrateam allocation rules we consider are not contests given that the allocation does not depend on individual efforts.[2] The literature on how to split a divisible prize to incentivize individual players typically concludes that relying on one prize only generates more incentives than having multiple smaller prizes (see Sisak 2009 for a survey). For example, Moldovanu and Sela (2001) consider the way to design prize allocation in a contest. They show that it is optimal to allocate all the money to a big prize rather than to several smaller prizes. The tournament literature also considers the case of multiple prizes; see for instance Nalebuff and Stiglitz (1983). One major advantage of our approach is its analytical tractability that allows for closed-form solutions.

We also contribute to this literature by extending it to the case of multiple *indivisible* prizes. Crutzen

[2]Crutzen, Flamand and Sahuguet (2020) analyze a model in which the allocation of prizes within teams is itself a contest.

et al. (2020) introduced the model of team contest with the allocation of prizes between teams following a multinomial distribution based on relative team efforts. We extend the model by allowing heterogeneity among team members in terms of ability and cost of effort, pin down conditions that guarantee existence of a unique equilibrium and fully characterize it. Crutzen et al. (2020) studied some real-world allocation rules and compare their effectiveness in terms of providing incentives to candidates. In this paper, we characterize the optimal allocation rule. We show that it usually provides non-montonic incentives to candidates, in the sense that a candidate's probability of winning a seat can decrease as his party wins more seats. This means that the list rule and the egalitarian rule studied in Crutzen et al. (2020) are genreally not optimal when we allow for non-monotonic allocation of seats. On the more applied side, our paper contributes to a recent literature that analyzes the design of lists in proportional electoral systems.[3] We relate our findings to that literature at the end of Section 5.

3 The Model

3.1 Effort, Objectives and Timing of the Contest

J teams are competing in a contest with n identical prizes. Team $j = 1, ..., J$ is composed of n_j (productive) members. Members differ in their ability $a_{ij} \geq 0$, and their cost of effort parameter $c_{ij} > 0$. We denote by N_j the set of (productive) team j members ($|N_j| = n_j$). If $n_j < n$, the other $n - n_j$ team members are unproductive dummy members of team j. They don't exert effort, and we assume that they may win prizes only when the number of prizes won by team j exceeds n_j. Teams compete for n indivisible identical prizes. The number of prizes won by team j is a random variable. We assume that team j's probability of winning a given prize follows a binomial distribution with parameter p_j. Parameter p_j is the outcome of a generalized Tullock contest with $\gamma \in (0, 1]$, based on the aggregate effort E_j of each team:

$$p_j = \frac{E_j^\gamma}{E_j^\gamma + \sum_{\ell \neq j} E_\ell^\gamma}.$$

The case $\gamma = 1$ corresponds to the classic Tullock contest success function (see Tullock 1980).

[3] See for instance Crutzen et al. (2022), Cox et al. (2021), and Buisseret et al. (2022).

We assume team j's aggregate effort E_j to be:

$$E_j = \sum_{i=1}^{n_j} a_{ij} e_{ij}.$$

where e_{ij} is the effort of member i in team j. Every player in team j has the following convex cost of effort function:

$$C_{ij} = \frac{1}{\beta} c_{ij} e_{ij}^{\beta},$$

with $\beta > 1$ and $c_{ij} > 0$ for all ij.

Prizes are awarded from independent draws of the distribution $(p_1, ..., p_J)$. Thus, team j's probability of winning k prizes follows a multinomial distribution and is equal to:

$$P_j^k = C(n,k) p_j^k \left(1 - p_j\right)^{n-k}.$$

Before members exert effort, teams decide on how to allocate prizes that will be won in the contest. Let $\mathcal{S}(k) \equiv \{S \subseteq N_j : |S| = k\}$. A **(stochastic) prize assignment rule** is a list of functions $q_j = (q_j^k)_{k=1}^n$ such that $q_j^k : \mathcal{S}(k) \to [0,1]$ and $\sum_{S \in \mathcal{S}(k)} q_j^k(S) = 1$ for all $k = 1, ..., n$. We assume that for all $k \geq n_j$, $q_j^k(S) > 0$ implies $N_j \subseteq S$: that is, the productive members of team j will get a prize for sure as long as the team gets n_j prizes or more. A prize assignment rule assigns probabilities to which subset of k team members win a prize when k prizes are won in the contest. Member i of team j has the following benefit function

$$B_{ij} = V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) P_j^k(p_j),$$

where $\mathcal{S}_i(k) = \{S \in \mathcal{S}(k) : i \in S\}$.

Knowing their team's allocation rule, player ij chooses effort to maximize:

$$U_{ij} = B_{ij} - C_{ij} = V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) P_j^k(p_j) - \frac{1}{\beta} c_{ij} e_{ij}^{\beta}.$$

Players' efforts $(e_{ij})_{ij \in N_j}$ determine team j's winning probability $P_j^k(p_j)$ via their impact on E_j and thus p_j. Following Nitzan and Ueda (2011), we assume that team members observe only their own team's prize assignment rule, but not those of other teams.

The timeline and the information structure of the two-stage game is as follows:

1. Stage 1: Teams choose a prize assignment rule (or a randomization over prize assignment rules) to maximize the expected number of prizes they win.

2. Stage 2: Observing their own team's assignment rule only, all team members choose their effort simultaneously and independently, to maximize their expected payoff.

Following Nitzan and Ueda (2011), we focus on a **perfect Bayesian equilibrium with own-action-independent beliefs**. Even if team j's manager deviates from the equilibrium prize assignment rule q_j, the members of team j react to it without changing the expectations on the choices made in the other teams; they believe that the deviation does not correlate with the events outside of their own team.[4] An equilibrium consists in an assignment rule profile, and stage 2 equilibrium efforts as a function of the assignment rule profile, $e(q) = (e_{ij}(q))$. An equilibrium gives rise to an equilibrium function $p^* = (p_1^*, ..., p_J^*)$ such that for all $j = 1, ..., J$, $p^*(q)$ assigns a winning probability profile to each subgame $q = (q_j)_{j \in J}$.

Equilibrium requires that the assignment rule maximizes the teams' expected number of prizes won given the other assignment rules chosen by other teams, and the stage-2 effort choices. A player's effort in stage 2 maximizes his expected payoff for any assignment rule profile, given the effort strategy of other players.

3.2 Prize-Assignment Rule and Prize-Assignment Matrix

An alternative way to describe prize-assignment rules is to use an **assignment matrix**. Let $r_{ij} = (r_{ij}^k)_{k=1}^n$ be a vector where r_{ij}^k denotes the probability of team member i's winning a prize when his team wins k

[4]Nitzan and Ueda (2011) call it a pure strategy perfect Bayesian equilibrium with a "no-signaling-what-you-don't-know" condition (see also Fudenberg and Tirole 1991, page 332).

prizes for each $k = 1, ..., n$. We can then write an $n \times n$ assignment matrix R_j as:

$$R_j = \begin{pmatrix} r_{1j}^1 & \cdots & r_{1j}^k & \cdots & r_{1j}^n \\ \vdots & \ddots & \vdots & & \vdots \\ r_{ij}^1 & \cdots & r_{ij}^k & \cdots & r_{ij}^n \\ \vdots & & \vdots & \ddots & \vdots \\ r_{nj}^1 & \cdots & r_{nj}^k & \cdots & r_{nj}^n \end{pmatrix}$$

with $r_{ij}^k \in [0, 1]$ and $\sum_{i=1}^n r_{ij}^k = k$ for all $k = 1, ..., n$. Since players $i = n_j + 1, ..., n$ are dummy players who may win the prizes only when $k > n_j$, we assume $r_{ij}^k = 1$ for all $k \geq n_j$ and all $i \in N_j$, and $r_{ij}^k = 1$ if and only if $i \leq k$ for $i = n_j + 1, ..., n$. A general prize-assignment rule can be represented by a matrix R_j by setting $r_{ij}^k = \sum_{S \in \mathcal{S}(k)} q_j^k(S)$ for each $i = 1, ..., n$ and $k = 1, ..., n$. However, it is not clear whether or not every R_j can be described by a general prize-assignment rule—can the entire space of R_js be spanned by general prize-assignment rules? The following lemma provides a positive answer.

Lemma 1. Any $n \times n$ assignment matrix R_j such that (i) $r_{ij}^k \in [0, 1]$ for all $i = 1, ..., n$, and all $k = 1, ..., n$, and (ii) $\sum_{i=1}^n r_{ij}^k = k$ for all $k = 1, ..., n$, can be achieved by some allocation rule $q_j : S \to [0, 1]$ with $\sum_{S \in \mathcal{S}(k)} q_j^k(S) = k$ for all $k = 1, ..., n$.

Remark 1. *In the matching literature, random assignments of indivisible goods often use the property known as the Birkhoff-von Neumann theorem (Birkhoff 1946, and von Neumann 1953): any bistochastic matrix can be written as a convex combination of permutation matrices (see, for example, Bogomolnaia and Moulin 2001). Our lemma may appear to be related to this theorem. However, in our model, indivisible prizes are homogenous and the number of them is stochastic. We do not think that there exists a formal relationship between the two models.*

Thus, without loss of generality, we use assignment matrices in the rest of the paper. The assumption that dummy members can win a prize only after all productive members of the team win a prize (if $k \geq n_j$, $q_j^k(S) > 0$ implies $N_j \subseteq S$) can be written as $r_{ij}^k = 1$ for all $i \in N_j$ and all $k \geq n_j$.

4 Equilibrium Efforts (Stage 2)

We consider a given profile of assignment rules $R_j = (r_{ij}^k)_{i,k=1,\ldots,n}$, and solve for the equilibrium efforts in stage 2.

The benefit function of member i of team j is:

$$\begin{aligned}
B_{ij} &= V\sum_{k=1}^{n} r_{ij}^k P^k(p_j) \\
&= V\sum_{k=1}^{n_j} r_{ij}^k P^k(p_j) + V\sum_{k=n_j+1}^{n} P^k(p_j) \\
&= V\sum_{k=1}^{n_j} r_{ij}^k P^k(p_j) + V\left(1-\sum_{k=1}^{n_j} P^k(p_j)\right) \\
&= V\sum_{k=1}^{n_j} r_{ij}^k P^k(p_j) + V\left\{1-(n-n_j)\,C(n,n_j)\int_0^{1-p_j} t^{n-n_j-1}(1-t)^{n_j}dt\right\}.
\end{aligned}$$

The impact of an increase in p_j on $P^k(p_j)$ is not straightforward. P_j^k does not always increase with an increase in p_j. Differentiating P_j^k and $\sum_{k=n_j+1}^{n} P_j^k(p_j)$ with respect to p_j, we obtain

$$\begin{aligned}
\frac{dP_j^k}{dp_j} &= C(n,k)p_j^{k-1}(1-p_j)^{n-k-1}(k-np_j) \quad \text{for } k=1,...,n-1, \qquad (1)\\
\frac{dP_j^k}{dp_j} &= np_j^{n-1} \text{ if } k=n,
\end{aligned}$$

and

$$\frac{d\sum_{k=n_j+1}^{n} P_j^k(p_j)}{dp_j} = (n-n_j)\,C(n,n_j)p_j^{n-n_j-1}(1-p_j)^{n_j}.$$

The probability of party j's winning exactly k prizes decreases with an increase in p_i for $k < k^* = \lfloor np_i \rfloor + 1$ (or $k < np_i$), and increases for $k \geq k^*$ (or $k > np_j$).

We also have

$$\frac{\partial E_j}{\partial e_{ij}} = a_{ij},$$

and

$$\frac{\partial p_j}{\partial E_j} = \frac{\gamma E_j^{\gamma-1}\left(E_j^\gamma+\sum_{k\neq j}E_k^\gamma\right)-\gamma E_j^{\gamma-1}E_j^\gamma}{\left(E_j^\gamma+\sum_{k\neq j}E_k^\gamma\right)^2} = \frac{\gamma}{E_j}p_j\left(1-p_j\right),$$

resulting in

$$\frac{\partial p_j}{\partial e_{ij}} = \frac{\gamma a_{ij}}{E_j}p_j\left(1-p_j\right). \qquad (2)$$

Taking the derivative of $B_{ij} = V \sum_{k=1}^{n} r_j^k P^k(p_j)$ with respect to e_{ij}, using (1), (2), and $\frac{\partial E_j}{\partial e_{ij}} = a_{ij}$, yields:

$$\begin{aligned}
\frac{\partial B_{ij}}{\partial e_{ij}} &= V \sum_{k=1}^{n_j} r_i^k \frac{dP^k}{dp_j} \frac{\partial p_j}{\partial e_{ij}} + V\left(n - n_j\right) C(n, n_j) p_j^{n-k-1} (1 - p_j)^k \frac{\partial p_j}{\partial e_{ij}} \\
&= \frac{\gamma V}{E_j} \sum_{k=1}^{n_j} r_i^k C(n,k) \left\{ k p_j^{k-1} \left(1 - p_j\right)^{n-k} - \left(n - k\right) p_j^k \left(1 - p_j\right)^{n-k-1} \right\} \left(1 - p_j\right) p_j a_{ij} \\
&\quad + \frac{\gamma V}{E_j} \left(n - n_j\right) C(n, n_j) p_j^{n-n_j} (1 - p_j)^{n_j+1} a_{ij} \\
&= \frac{\gamma V}{E_j} a_{ij} \left\{ \sum_{k=1}^{n_j} r_i^k C(n,k) p_j^k \left(1 - p_j\right)^{n-k} \left(k - n p_j\right) + \left(n - n_j\right) C(n, n_j) p_j^{n-n_j} \left(1 - p_j\right)^{n_j+1} \right\} \\
&= \frac{\gamma V}{E_j} a_{ij} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\}
\end{aligned}$$

where

$$\begin{aligned}
\mu^k(p_j) &\equiv \frac{dP^k(p_j)}{dp_j} p_j \left(1 - p_j\right) \\
&= C(n,k) p_j^k \left(1 - p_j\right)^{n-k} \left(k - n p_j\right)
\end{aligned}$$

and

$$\begin{aligned}
\nu^{n_j}(p_j) &\equiv \frac{d \sum_{k=n_j+1}^{n} P_j^k(p_j)}{dp_j} p_j \left(1 - p_j\right) \\
&= \left(n - n_j\right) C(n, n_j) p_j^{n-n_j} \left(1 - p_j\right)^{n_j+1}
\end{aligned}$$

denote the impacts of an increase in p_j on P^k and $\sum_{k=n_j+1}^{n} P_j^k(p_j)$, respectively.

Thus, *tentatively* assuming an interior solution, the first order condition reads:

$$\frac{\partial B_{ij}}{\partial e_{ij}} - C'_{ij}(e_{ij}) = \frac{\gamma V}{E_j} a_{ij} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} - c_{ij} e_{ij}^{\beta-1} = 0. \quad (3)$$

Since $C''_{ij}(e) > 0$ holds for all $e > 0$, and $\lim_{e \to 0} C'_{ij}(e) = 0$ and $\lim_{e \to \infty} C'_{ij}(e) = \infty$, there is a unique solution for the above problem:

$$e_{ij}^{\beta-1} = \left[\frac{1}{c_{ij}} \frac{\gamma V}{E_j} a_{ij} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} \right]$$

or

$$e_{ij} = \left[\frac{1}{c_{ij}} \frac{\gamma V}{E_j} a_{ij} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} \right]^{\frac{1}{\beta-1}}.$$

Summing team members' efforts up, we obtain

$$E_j = \sum_{h=1}^{n_j} a_{hj} \left[\frac{1}{c_{hj}} \frac{\gamma V}{E_j} a_{ij} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} \right]^{\frac{1}{\beta-1}}$$

This interior solution for e_{ij} only makes sense when $\sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) > 0$. This condition states that member j's probability to win a prize is positively affected by an increase in p_j. If the sign of this expression is negative, then player i's marginal effort worsens her payoff and $e_{ij} = 0$ must hold. Thus, we have the following general formula:

$$E_j = \sum_{h=1}^{n_j} a_{hj} \left[\frac{1}{c_{hj}} \frac{\gamma V}{E_j} a_{ij} \max \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j), 0 \right\} \right]^{\frac{1}{\beta-1}}$$

The first-order conditions identify optimal effort only if the second-order conditions are satisfied. Unfortunately, the second order conditions are not always satisfied for all players. To get a fully general condition on exogenous parameters, we need to assume a small value of parameter γ. However, the following lemma shows that the second-order conditions are satisfied for any arbitrary assignment rule r_i for reasonable parameter values.

Lemma 2 (Second-order conditions). Let $\theta_{ij} = \frac{a_{ij} e_{ij}}{E_j}$ denote player i's effort contribution share in team j. The second order condition for player i is satisfied if $\gamma \leq \frac{1 + \frac{(\beta-1)}{\theta_{ij}}}{\max\{n_j - np_j, n - n_j - np_j\}}$.

Remark 2. *Consider the following example with the following parameter values: $n = 20$, $n_j = 10$, $p_j = \frac{1}{4}$, and $\beta = 2$. In the worst-case scenario, when there is no other productive player in the team ($\theta_{ij} = 1$), the above condition reads $\gamma \leq \frac{2}{5}$. Considering the standard Tullock contest case $\gamma = 1$ the above condition is satisfied if $\theta_{ij} \leq \frac{1}{4}$. In this example, the expected number of prizes won is $np_j = 5$, the number of active team members is $n_j = 10$ considering a highest effort share of $\max_i \theta_{ij} = \frac{1}{4}$ appears reasonable in most cases. Extreme heterogenity in abilities could of course lead to one player having a larger effort contribution share.*

Although the condition for second-order conditions to be satisfied does not appear too restrictive, the sufficient conditions in Proposition 1 are not expressed in terms of exogenous parameters of the model (p_j and

θ_{ij} are endogenous variables determined in equilibrium). Thus, for the sake of rigor, we impose the following simple but restrictive sufficient condition that works for any team member profile and any assignment rule by setting $\theta_{ij} = 1$ and $p_j = 0$.[5]

Assumption 1. The contest success function is sufficiently concave $\gamma \leq \frac{\beta}{\max\{n_j, n-n_j\}}$.

We can now state the following result:

Proposition 1 (Effort choice). Under assumption 1, for each $p_j \in (0,1)$, and each assignment matrix $R_j = (r_{ij}^k)_{i,k=1,\ldots,n}$, there is a unique Nash equilibrium, with effort vector $e_i^*(p_j, R_j) \equiv (e_{ij}^*(p_j, R_j))_{i \in N_j}$: for all $i \in N_j$,

$$e_{ij}^*(p_j, R_j) = \frac{\left(\frac{a_{ij}^\beta}{c_{ij}} \rho_{ij}(p_j, R_j)\right)^{\frac{1}{\beta-1}}}{a_{ij} \sum_{h=1}^n \left(\frac{a_{hj}^\beta}{c_{hj}} \rho_{hj}(p_j, R_j)\right)^{\frac{1}{\beta-1}}} E_j(p_j, R_j).$$

Team j's aggregate effort is

$$E_j(p_j, R_j) = (\gamma V)^{\frac{1}{\beta(\beta-1)}} \left(\sum_{h=1}^{n_j} \left(\frac{a_{ij}^\beta}{c_{ij}} \rho_{ij}(p_j)\right)^{\frac{1}{\beta-1}}\right)^{\frac{1}{\beta}},$$

where $\rho_{ij}(p_j, R_j) = \max\left\{\sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j), 0\right\}$.

We now prove the existence and uniqueness of equilibrium for given assignment rules. As E_j depends only on p_j ($E_j = E_j(p_j)$), we consider the following mapping $f(p)$, with $p = (p_1, ..., p_J)$ and

$$f_j(p) = \frac{E_j^\gamma(p_j)}{\sum_{\ell=1}^J E_\ell^\gamma(p_k)}$$

for all $j = 1, ..., J$. Then $f(p) = (f_1(p), ..., f_J(p))$ is a fixed point mapping from simplex $\Delta^J \equiv \left\{p \in \mathbb{R}_+^J : \sum_{k=1}^J p_k = 1\right\}$ to itself, which is a continuous function. Since Δ^J is nonempty, compact, and convex, and $f : \Delta^J \to \Delta^J$ is a continuous function, there exists a fixed point $p^* = f(p^*)$ by Brouwer's fixed point theorem.

[5] The most difficult second-order conditions to satisfy comes from the following situation. Suppose that $n_j = n$ (thus $\nu^{n_j}(p_j) = 0$) and player i can win a prize only when team j wins all the prizes $r_{ij}^k = 1$, that is, if and only if $k = n$. Moreover, this player is the only active effort contributor in team i. In this case, $P_j^n(p_j) = C(n,n)p_j^n$ is very strongly convex function, and we need to set $\gamma \leq \frac{\beta}{n}$ to cancel out this convexity to satisfy concave objective function for such player ij.

Proposition 2 (Existence). Under assumption 1, there exists a Nash equilibrium for any assignment matrix profile $(R_j)_{j=1}^J$.

We now turn to uniqueness. We consider the equilibrium relationship between p_j and E_j. Recall that

$$E_j(p_j, R_j) = (\gamma V)^{\frac{1}{\beta(\beta-1)}} \left(\sum_{i=1}^{n_j} \left(\frac{a_{ij}^\beta}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}} \right)^{\frac{1}{\beta}}.$$

The equilibrium condition is summarized by the following system of n equations:

$$p^* = f(p^*).$$

We will drop the last equation or our equilibrium condition by setting $p_J = 1 - \sum_{j=1}^{J-1} p_j$ (since $\sum_{j=1}^{J} p_j = 1$), and we totally differentiate the system. In the proof of Proposition 3 in the Appendixc, we show that the determinant of the system is strictly positive. Then, we can use the index theorem to prove uniqueness of equilibrium.

Proposition 3 (Uniqueness) Under assumption 1, for any assignment matrix profile $R = (R_j)_{j=1}^J$, there is a unique Nash equilibrium characterized by an effort vector $e_i^*(R_j)$ and a winning probability vector $p^*(R) = \left(p_j^*(R)\right)_{j=1}^J$. Moreover, $p^*(R)$ is a continuous function.

5 Equilibrium of Choice of Assignment Rule (Stage 1)

5.1 Existence of Equilibrium

We have characterized the second stage equilibrium in Proposition 1, its existence in Proposition 2 and its uniqueness in Proposition 3. We now turn attention to the choice of assignment rule in stage 1.

Since $(P^k(p_j'))_{k=1}^n$ first-order stochastically dominates $(P^k(p_j))_{k=1}^n$ for $p_j' > p_j$, team j should maximize p_j. As $p_j = \frac{E_j^\gamma}{E_1^\gamma + ... + E_J^\gamma}$, team j should choose rule R_j to maximize E_j given E_{-j}. However, $p = (p_1, ..., p_j, ..., p_J)$ is actually determined in the interaction with other teams, so we need to check that maximizing E_j is equivalent to maximizing the number of prizes won.

We consider the following comparative static exercise: we increase r_{ij}^k by $\Delta_j > 0$ and decrease in r_{hj}^k by Δ_j for some $i, h = 1, ..., n$ with $i \neq h$ and $k \in \{1, ..., n\}$ with $\frac{a_{ij}^\beta}{c_{i1}} > \frac{a_{hj}^\beta}{c_{h1}}$ and $\mu^k(p_j) > 0$. This change in the assignment rule is assumed to increase aggregate effort E_j for given p_j. We show that this change also increases its winning probability.[6]

Proposition 4. Under assumption 1, the allocation rules R_j that maximizes

$$E_j(p_j, R_j) = (\gamma V)^{\frac{1}{\beta(\beta-1)}} \left(\sum_{h=1}^{n_j} \left(\frac{a_{ij}^\beta}{c_{ij}} \max\left\{ \sum_{k=1}^{n_j} r_{hj}^k \mu^k(p_j) + \nu^{n_j}(p_j), 0 \right\} \right)^{\frac{1}{\beta-1}} \right)^{\frac{1}{\beta}}$$

also maximizes p_j.

Now, we can prove the existence of an equilibrium of the two-stage game. In stage 1, each team j selects an assignment matrix R_j to maximize its aggregate effort E_j:

$$E_j(R_j, p_j^*) = (\gamma V)^{\frac{1}{\beta(\beta-1)}} \left[\sum_{h=1}^{n_j} \alpha_{ij} \left(\max\left\{ \sum_{k=1}^{n_j} r_{ij}^k \mu^k(p_j^*) + \nu^{n_j}(p_j^*), 0 \right\} \right)^{\frac{1}{\beta-1}} \right]^{\frac{1}{\beta}}$$

where $\alpha_{ij} \equiv a_{ij}^{\frac{\beta}{\beta-1}} / c_{ij}^{\frac{1}{\beta-1}}$ is player i's effectiveness in making effort. Let $\mathcal{R}$ be the collection of all possible allocation rules R. From Proposition 4, a winning probability maximizing team j solves the following problem:

$$\max_{R_j \in \mathcal{R}} E_j(p_j^*, R_j) \text{ s.t. } \sum_{i=1}^{n} r_{ij}^k = k \text{ for all } k = 1, ..., n. \tag{4}$$

We can prove the existence of an equilibrium, using a fixed point mapping argument.

Theorem 1. Under assumption 1, there exists a perfect Bayesian equilibrium with own action independent beliefs $(R^*; p^*)$, where $R^* \in \mathcal{R}$ is a stage 1 equilibrium assignment matrix profile, R_{-j}^* is the belief for all team $j \in J$, and $p^* : R^J \to \Delta^J$ is a stage 2 winning probability mapping defined in Proposition 3.

[6]The issue at stake is reminisicent of some results in general equilibrium international trade theory. In some cases, economic growth can result in a country being worse off (see papers on immiserizing growth by Jonhson 1955 and Bhagwati 1958). In other cases, a country may gain by giving a transfer, that the receiver may lose, and that these two phenomena may appear at the same time in a three country model (see the paper on transfer paradox by Yano 1983). Our Proposition shows that our model is well-behaved: when a team designs an assignment rule to increase its aggregate effort, it results in an improvement in the team's winning probability.

Remark 3. *The first stage equilibrium matrices* $R^* = (R_1^*, ..., R_J^*)$ *requires some explanations. As we will see in the next section,* $E_j(p_j^*, R_j)$ *is concave (convex) in* R_j *if* $\beta \geq 2$ *(if* $1 < \beta \leq 2$*). Thus, when* E_j *is concave, even if* R_j^* *is stochastic, it is a best response to* R_{-j}^* *which maximizes* $E_j(p_j^*, R_j)$*. That is, team* j *announces a mixed strategy described by* R_j^* *to their members. In contrast, when* E_j *is convex, if* R_j^* *is stochastic,* R_j^* *itself is not a best response, since the best response must be a deterministic matrix due to convex payoff function. In this case,* R_j^* *must be a convex combination of a set of deterministic matrices* $R_j^{d_1}, ..., R_j^{d_L}$*, each of which is a best response to* R_{-j}^**. The team announces to its members one of them randomly. Thus, when* $1 < \beta \leq 2$*, randomization* R_j^* *is introduced only to find a Nash equilibrium played by* J *teams—a fixed point* $R^* = (R_1^*, ..., R_J^*)$*.*

Remark 4. *Theorem 1 holds even if we impose constraints on parties' strategy spaces (as long as the set of admissible matrices* $\mathcal{R}$ *is a nonempty, compact, and convex set). For example, we can restrict team* j *to use only list rules as the support of their mixed strategy. Theorem 1 still holds, and team* j *uses the optimal list rule in equilibrium.*[7]

5.2 Optimal Prize-Assignment Matrix

We now characterize these optimal assignment matrices R_j.

As $(\gamma V)^{\frac{1}{\beta(\beta-1)}} > 0$ is a positive constant and $\frac{1}{\beta} > 0$, team j's maximization problem (4) can be simplified to:

$$\max_{R_j \in \mathcal{R}} \pi_j(p_j^*, R_j) = \max_{(r_{ij}^k)_{i \in N_j, k=1,...,n}} \sum_{i=1}^{n_j} \alpha_{ij} \left(\max \left\{ \sum_{k=1}^{n_j} r_{ij}^k \mu^k(p_j^*) + \nu^{n_j}(p_j^*), 0 \right\} \right)^{\frac{1}{\beta-1}} \quad \text{s.t.} \quad \sum_{i=1}^{n} r_{ij}^k = k, \forall k = 1, ..., n$$

where $\mu^k(p_j^*) \equiv C(n,k) p_j^{*k} \left(1 - p_j^*\right)^{n-k} \left(k - np_j^*\right)$.

Note that $\mu^k(p_j^*) \equiv C(n,k) p_j^{*k} \left(1 - p_j^*\right)^{n-k} \left(k - np_j^*\right)$ changes its sign only once at $k^* \equiv \lfloor np_j^* \rfloor + 1$, which is the smallest integer that exceeds np_j^*. We have $\mu^k(p_j^*) < 0$ for all $k < k^*$, and $\mu^k(p_j^*) > 0$ for all $k \geq k^*$ ($\mu^{k^*}(p_j^*) = 0$ if $k^* = np_j^*$).

[7] We can also assume that other teams use different rules—an arbitrary rule or the egalitarian rule (every team member wins a prize with equal probability no matter what the number of prizes won is). Theorem 1 still applies.

There are two cases to consider depending on the value of β: if $1 < \beta \leq 2$, $\frac{1}{\beta-1} \geq 1$ holds, and π_j is convex in R_j, while if $\beta > 2$ then $\frac{1}{\beta-1} < 1$ holds and π_j is concave on R_j. Regarding $\mu^k(p_j^*)$s as constants around the optimum, the former case requires to assign players with highest effectiveness parameters the highest total weights. In contrast, in the latter case, mixing is preferable. These two cases correspond to concave and convex marginal effort cost functions, respectively (see Esteban and Ray 2001, Nitzan and Ueda 2014, and Trevisan 2020).

5.3 Convex Objective Function ($1 < \beta \leq 2$)

In this case, the maximization problem requires to assign players with highest effectiveness parameters to the highest total weights. Thus, we consider *deterministic* assignment rules R_j with $r_{ij}^k \in \{0, 1\}$ for all $i, k = 1, ..., n$.[8] Relabel team members using their "effectiveness" in an descending order: $\alpha_{1j} \geq \alpha_{2j} \geq ... \geq \alpha_{n_i j}$. The power of the parenthesis $\frac{1}{\beta-1}$ in the objective function satisfies $\frac{1}{\beta-1} \geq 1$. In that case, putting more weight r on a player with a higher effectiveness when $\mu^k(p_j^*) > 0$ increases overall effort. In order to maximize $\pi_j(R_j, p_j) = \sum_{i=1}^{n} \alpha_{ij} \left(\max \left\{ \sum_{k=1}^{n_j} r_{ij}^k \mu^k(p_j^*) + \nu^{n_j}(p_j^*), 0 \right\} \right)^{\frac{1}{\beta-1}}$ around the equilibrium winning probability $p_j = p_j^*$, team j assigns the highest of the sum of weights to $i = 1$, and the second highest to $i = 2$, and so on. Such an assignment rule R_j^* is the best response around the equilibrium p_j^*, thus, R_j^* is the equilibrium assignment rule that is in the best response to R_{-j}^* (the linearly approximated best response is the best response to the original problem in the equilibrium). Notice that in this problem, $\nu^{n_j}(p_j^*)$ is common to all productive team members, and it will be irrelevant to the choice of optimal assignment matrix. *Thus, from here on, we assume that $n_j = n$ without loss of generality.*

We first illustrate the optimal assignment rules in the following examples (recall that $k^* \equiv \lfloor np_j^* \rfloor + 1$).

Example 1. Suppose $n(= n_j) = 7$. We consider two cases: $k^* = 3$ and $k^* = 5$.

We need to choose k players to win when team j wins k prizes. Starting from the most effective player 1, we decide sequentially whether or not a player should be awarded a prize for each number of winning prizes k. Let $\kappa(i) = (\kappa_1(i), ..., \kappa_k(i), ..., \kappa_n(i))$ be the vector describing the number of prizes left to be assigned

[8] See Remark 3.

when team member i is considered. For example, $\kappa_k(i)$ tells us how many prizes are left to be assigned when team member i is considered and the team wins k prizes. Thus, initially, we have $\kappa(1) = (1, 2, 3, 4, 5, 6, 7)$ seats to assign in the case of team j's winning $k = 1, ..., 7$ prizes.

We start with $k^* = 3$. This case corresponds to a low value of p_j; team j is not a favorite in the election. We derive the optimal assignment matrix R_3^* sequentially. Initially, we have $\kappa(1) = (1, 2, 3, 4, 5, 6, 7)$. We start with the most effective player 1. As $\mu^k(p_j^*) \gtreqless 0$ if and only if $k \gtreqless np_j^*$ or $k \gtreqless k^*$ ($k^* \equiv \lfloor np_j^* \rfloor + 1$), in order to place the heaviest weights $\sum_{k=k^*}^{7} \mu^k(p_j^*)$ on player 1, player 1 should win a prize if and only if team j wins $k^* = 3$ or more seats, since $\mu^k(p_j^*) > 0$ for $k \geq 3$, and $\mu^1(p_j^*) < 0$ and $\mu^2(p_j^*) < 0$. Thus, we have $r_{1j} = (r_{1j}^k)_{k=1}^7 = (0, 0, 1, 1, 1, 1, 1)$. Then, when player 2's turn comes, the number of left-over prizes is $\kappa(2) = (1, 2, 2, 3, 4, 5, 6)$. So, player 2 still can get the same prize vector $r_{2j} = r_{1j} = (0, 0, 1, 1, 1, 1, 1)$, and $\kappa(3) = (1, 2, 1, 2, 3, 4, 5)$ remains. Still, player 3 can get $r_{3j} = r_{1j} = (0, 0, 1, 1, 1, 1, 1)$. However, when player 4's turn comes, $\kappa(4) = (1, 2, 0, 1, 2, 3, 4)$. Therefore player 4 cannot get a prize when $k = 3$. So, the highest feasible weight for this player is achieved by $r_{4j} = (0, 0, 0, 1, 1, 1, 1)$, leaving $\kappa(5) = (1, 2, 0, 0, 1, 2, 3)$. Thus, player 5 gets $r_{4j} = (0, 0, 0, 0, 1, 1, 1)$. When player 6's turn comes, $\kappa(6) = (1, 2, 0, 0, 0, 1, 2)$, but only 2 members (players 6 and 7) are left to be considered. Thus, player 6 needs to get a prize for $k = 2$ in addition to positive weights $k = 6, 7$. Otherwise, when $k = 2$, two prizes will not be allocated. The following table summarizes $\kappa(i)$, $\chi(i)$ (the number of players to be considered), and r_{ij} at each step i:

i	leftover prizes $\kappa(i)$	$\chi(i)$	assigment r_{ij}
1	1 2 3 4 5 6 7	7	0 0 1 1 1 1 1
2	1 2 2 3 4 5 6	6	0 0 1 1 1 1 1
3	1 2 1 2 3 4 5	5	0 0 1 1 1 1 1
4	1 2 0 1 2 3 4	4	0 0 0 1 1 1 1
5	1 2 0 0 1 2 3	3	0 0 0 0 1 1 1
6	1 2 0 0 0 1 2	2	0 1 0 0 0 1 1
7	1 1 0 0 0 0 1	1	1 1 0 0 0 0 1

As a result, the optimal assignment matrix is as follows:

$$R_j^{3*} = \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Bold lettered 1s provide positive incentives to make effort, while normal lettered 1s provide negative incentive to make effort. R_j^{3*} gives incentives to high ability members to exert effort, while lower ability team members get less or no incentive to exert effort.[9]

We now turn to the case $k^* = 5$. With a high value of p_j, team j is the favorite in the election. With $k^* = 5$, $\mu^k(p_j^*) > 0$ only for $k \geq 5$. Thus, in order to maximize the weight of the highest member 1, she should win a prize as long as the number of prizes won is three or more: $r_{1j} = (0, 0, 0, 0, 1, 1, 1)$. The following table summarizes $\kappa(i)$, $\chi(i)$ (the number of players to be considered), and r_{ij} at each step i:

[9] If there are dummy players and they are allowed to get a prize even when $k \leq n_j$, then they should get prizes when $k < k^*$ for the same reason.

i	leftover prizes $\kappa(i)$	$\chi(i)$	assigment r_{ij}
1	1 2 3 4 5 6 7	7	0 0 0 0 1 1 1
2	1 2 3 4 4 5 6	6	0 0 0 0 1 1 1
3	1 2 3 4 3 4 5	5	0 0 0 0 1 1 1
4	1 2 3 4 2 3 4	4	0 0 0 1 1 1 1
5	1 2 3 3 1 2 3	3	0 0 1 1 1 1 1
6	1 2 2 2 0 1 2	2	0 1 1 1 0 1 1
7	1 1 1 1 0 0 1	1	1 1 1 1 0 0 1

The same procedure as the above creates the following optimal assignment matrix:

$$R_j^{5*} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

The highest ability members 1, 2, and 3 win a prize only when their team wins 5 or more prizes. The team wants the highest ability members to exert effort to win a prize.□

Formally, in order to describe the optimal (deterministic) mechanism, we use an algorithm. For each $k = 1, ..., n$, the team needs to allocate k prizes in total. We will assign prizes in a descending order starting from the highest effectiveness team member $i = 1$. Recall that for each number of winning prizes $k = 1, ..., n$, $\kappa(i) = (\kappa_1(i), ..., \kappa_k(i), ..., \kappa_n(i))$ is the number of prizes left to be assigned when team member i is considered, and that $\nu(i) = n - i + 1$ is the number of players left to be considered when member i's turn comes. Obviously, before starting ($i = 1$), $\kappa(1) = (1, 2, ..., n)$ and $\chi(1) = n$.

As we are considering deterministic assignments, $\kappa_k(i) \leq k$ is a nonnegative integer for each k. Let $\mathcal{M}(i) \equiv \{k \in \{1, ..., n\} : \kappa_k(i) > 0\}$ be the set of cases k in which team member i can win a prize, let $\mathcal{L}(i) \equiv \{k \in \{1, ..., n\} : \kappa_k(i) = \chi(i)\}$ be the set of cases k in which team member i must win a prize (for feasibility: if not, k members of team j cannot win a prize when k prizes are won). Denote the effort-maximizing set of cases in which team member j wins a prize by $\zeta(i) \subseteq \{1, ..., k, ..., n\}$. Let $\kappa(i+1) = (\kappa_1(i+1), ..., \kappa_k(i+1), ..., \kappa_n(i+1))$ be such that $\kappa_k(i+1) = \kappa_k(i) - 1$ if $k \in \zeta(i)$, and $\kappa_k(i+1) = \kappa_k(i)$ otherwise. Initially, $\kappa(1) = (\kappa_1(1), ..., \kappa_k(1), ..., \kappa_n(1)) = (1, ..., k, ..., n)$, $\nu(1) = n$, $\mathcal{M}(1) \equiv \{1, ..., n\}$, and $\mathcal{L}(1) \equiv \{n\}$ hold. The reason for $\mathcal{L}(1) \equiv \{n\}$ is that if member 1 does not get a seat when n prizes are won, then it is not possible to distribute all n prizes. The set of cases for team member i to win a prize is defined by:

$$\zeta(i) = \arg \max_{\mathcal{L}(i) \subseteq K \subseteq \mathcal{M}(i)} \sum_{k \in K} \mu^k(p_j^*)$$

for $i = 1, ..., n$. This $\zeta(i)$ gives team member i the largest aggregate weights $\sum_{k \in \zeta(i)} \mu^k(p_j^*)$ available. The matrix is completed by setting $r_{ij}^k = 1$ if and only if $k \in \zeta(i)$ for all $i = 1, ..., n$ and all $k = 1, ..., n$.

Recall that $k^* \equiv \lfloor np_j^* \rfloor + 1$ is the smallest integer that exceeds np_j^*. By the definition of $\mu^k(p_j^*)$, $\zeta(1) = \{k^*, k^*+1, ..., n\}$ as this set collects all positive $\mu^k(p_j)$s without having any negative $\mu^k(p_j)$s. How about $\zeta(2)$? We have that $\zeta(2) = \{k^*, k^*+1, ..., n\}$ as long as $k^* \geq 2$ ($\kappa_{k^*}(2) \geq 1$), as $(2) \equiv \{1, ..., n\}$.

We need to consider two cases: (Case 1) $k^* \leq \frac{n+1}{2}$, and (Case 2) $k^* > \frac{n+1}{2}$.

(Case 1: $k^* \leq \frac{n+1}{2}$) We can assign the top k^* members to $\{k^*, k^*+1, ..., n\} = \zeta(1) = ... = \zeta(k^*)$. After that, as long as $i < n - k^* + 2$, we assign $\zeta(i) = \{i, i+1, ..., n\}$. When $i = n - k^* + 2$ comes, we assign $\zeta(i) = \{k^* - 1\} \cup \{i, i+1, ..., n\}$, and for $i = n - k^* + 3$, $\zeta(i) = \{k^* - 2, k^* - 1\} \cup \{i, i+1, ..., n\}$, and so on. When $i = n$, $\zeta(n) = \{1, ..., k^* - 1\} \cup \{n\}$.

(Case 2: $k^* > \frac{n+1}{2}$) In this case, we can only assign the top $n - k^*$ team members to $\{k^*, k^*+1, ..., n\} = \zeta(1) = \zeta(n - k^*)$. Since $\kappa_{k^*-1}(n - k^* + 1) = \kappa_n(n - k^* + 1) = n - (n - k^* + 1) + 1 = \nu(n - k^* + 1)$, $\zeta(n - k^* + 1) = \{k^* - 1, k^*, ..., n\}$. Similarly, up to $i = n - k^* + 1$, $\zeta(i) = \{n - i + 1, ..., n\}$ is assigned. After that $\zeta(i) = \{n - i + 1, ..., k^* - 1\} \cup \{i, ..., n\}$.

Note that if $\sum_{k \in \zeta(i)} \mu^k(p_j^*) \leq 0$, then $e_{ij} = 0$ holds. The outcome of this algorithm is an **effort-**

maximizing rule. This implies that the highest effectiveness team member wins a prize if and only if team j wins $k^* = \lfloor np_j^* \rfloor + 1$ prizes or more. That is, the highest effectiveness team member gets the same assignment in the optimal assignment rule and in the optimal list rule.

Theorem 2 (the optimal assignment rule: convex case). Suppose that $1 < \beta \leq 2$ holds. Then the optimal assignment rule is described by matrix R_j with $r_{ij}^k = 1$ if and only if $k \in \zeta(i)$ for all $i = 1, ..., n$ and all $k = 1, ..., n$.

In Theorem 1, equilibrium strategies are random matrices R_j^*s but Theorem 2 argues that the best response is deterministic rule when $1 < \beta \leq 2$ holds. As we have mentioned in Remark 3, this is not a contradiction. From Lemma 1, R_j^* is a convex combination of deterministic rules, thus, R_j^* can be regarded as a mix of pure strategies (deterministic rules) that achieve the same winning probability of team j, since $1 < \beta \leq 2$ implies that the best responses are in pure strategies. Thus, under this parameter restriction, Theorem 1 asserts the existence of mixed strategy equilibrium of a game in which teams announce a deterministic rule to their team members. Clearly, teams are is indifferent between the deterministic rules supporting R_j^*.

5.4 Concave Objective Function ($\beta \geq 2$)

With rapidly increasing marginal costs of effort, incentives should not be concentrated on a small set of team members.

Party j's maximization problem is

$$\max_{(r_i^k)} \sum_{i=1}^{n} \alpha_{ij} \left[\max\left(\sum_{k=1}^{n} r_{ij}^k \mu^k(p_j^*), 0 \right) \right]^{\frac{1}{\beta-1}} \quad \text{s.t.} \quad \begin{cases} \text{(i)} \displaystyle\sum_{i=1}^{n} r_i^k = k & \forall k \\ \\ \text{(ii)}\ 0 \leq r_i^k \leq 1 & \forall k, i \end{cases}$$

The optimal mechanism is the solution of the above problem when $\beta > 2$ holds. As k increases the set r_i^k will face stricter constraints (when $k = n$, $r_i^n = 1$ must hold: every team member needs to receive a prize). However, $\mu^k(p_j) = \frac{dP^k}{dp_j}(1 - p_j) p_j < 0$ for all $k < k^*$ and $\frac{dP^k}{dp_j}(1 - p_j) p_j > 0$ for all $k > k^*$, and what matters is just the weighted sum of the shares in the bracket in achieving the optimal allocation. Intuitively, there will be freedom using r^ks for low ks to achieve unequal allocations.

Although the above problem is a well-behaved Kuhn-Tucker constrained optimization problem, it is hard to obtain intuitive characterization of the optimum. Supposing that the sum of reward $\bar{R} = \sum_{k=1}^{n} k \frac{dP^k}{dp_j} (1 - p_j) p_j = \sum_{k=1}^{n} k\mu^k(p_j)$ can be allocated freely to team members according to their abilities, the optimal allocation is described by solving the following problem.

$$\max_{(R_i)_{i=1}^n} \sum_{i=1}^{n} \alpha_{ij} r_i^{\frac{1}{\beta-1}} \qquad s.t. \quad \sum_{i=1}^{n} r_i = \bar{R} = \sum_{k=1}^{n} k\mu^k(p_j)$$

The first order conditions generate the optimality conditions:

$$\frac{1}{\beta - 1}\alpha_{ij} r_i^{\frac{1}{\beta-1}-1} = \frac{1}{\beta - 1}\alpha_{hj} r_h^{\frac{1}{\beta-1}-1}$$

or

$$\frac{r_i}{r_h} = \left(\frac{\alpha_{ij}}{\alpha_{hj}}\right)^{\frac{\beta-1}{\beta-2}}$$

for all $i, h = 1, ..., n$.

Proposition 5. Suppose $\beta \geq 2$. Whenever feasible, the optimal assignment rule allocates the chances of team members to get a prize proportionally to their effectiveness (with power $\frac{\beta-1}{\beta-2}$).

This result is consistent with Proposition 5 in Crutzen, Flamand, and Sahuguet (2020) when the effort aggregation function is linear ($\sigma = 0$ in their CES function). When team members are homogeneous, $r_i = r_h$ holds for all $i, h = 1, ..., n$ when $\beta > 2(1 - \sigma)$. Thus $q_{ik} = \frac{k}{n}$ for all $i, k = 1, ..., n$, which generates the egalitarian rule analyzed in that paper.

Yet, the result in Proposition 5 does not take into account all the constraints in the maximization problem and also does not consider the fact that negative rewards do not lead to negative effort. We now consider some examples with 4 prizes and two identical teams to illustrate the optimal rule. We first look at the case of homogeneous team members and then consider the case with one high ability team member and three low ability members.

Example 2. Optimal rules with four prizes and all team members having the same effectiveness parameter.

At a symmetric equilibrium with $\alpha_{ij} = 1$, $\forall i$, we have that $p_1 = p_2 = 1/2$ which allows us to simplify the effort of a team to $E = \left(\sum_i \max\left(\left(\frac{V}{2^{n+1}} \sum_{k=1}^{n} r_{ik} C_k^n (2k-n)\right)^{\frac{1}{\beta-1}}, 0\right)\right)^{\frac{\beta-1}{\beta}}$. With $n = 4$, the objective function can be further simplified to $\sum_{i=1}^{4} \max\left((1 + 2(r_{i3} - r_{i1}))^{\frac{1}{\beta-1}}, 0\right)$.

When $\beta = 2$, the optimal allocation rule maximizes $\sum_{i=1}^{4} \max(1 + 2(r_{i3} - r_{i1}), 0)$. Under the egalitarian allocation rule (which satisfies Proposition 5 by giving all team members the same incentives) we get $\sum_{i=1}^{4} [(1 + 2(r_{i3} - r_{i1})] = 8$. Yet, this is not the optimal rule. Indeed, we can also consider to give negative incentives to some team members who, as a consequence, chooses not to exert effort. This frees incentive tokens that can be redistributed to other team member(s). It is easy to check that one optimal rule is:[10]

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

This allocation rule is non-monotonic in the sense that team member 1 wins a prize if the team wins one prize but does not win a prize if the team wins three prizes. With that rule, $\sum_{i=1}^{4} \max(1 + 2(r_{i3} - r_{i1}), 0) = 9 > 8$. Also, observe that members 2 to 4 are all treated equally (the weights in columns 2 and 4 do not matter for incentives): the entries in columns 1 and 3 are the same for these players. The incentives of all team members who exert strictly positive effort satisfy the condition in Proposition 5.

For $\beta > 2$, we can pin down the value of β below which it is optimal to depart from the egalitarian rule. To do this, one need only compare $\sum_{i=1}^{4}(1 + 2 * 1/2)^{\frac{1}{\beta-1}} = 4 * 2^{\frac{1}{\beta-1}}$ and $3 * (1 + 2 * 1)^{\frac{1}{\beta-1}} = 3 * 3^{\frac{1}{\beta-1}}$, where the effort of only three team members matter for the second team's output as, under the non-monotonic rule above, member 1 is inactive. Then, simple algebra implies that the egalitarian rule is suboptimal whenever $4 * 2^{\frac{1}{\beta-1}} < 3 * 3^{\frac{1}{\beta-1}} \iff \beta < \frac{\ln(3/2)+\ln(4/3)}{\ln(4/3)} \simeq 2.4$. The interpretation is straightforward. Non-monotonic incentives lead to more incentives in total but more unequal incentives. When β is close to 2, the first effect dominates as convexity is weak.

[10]This is not the only optimal rule, as optimality puts no constraints on the value the different r_{i2} can take on.

Example 3. Optimal rules with four prizes and one team member has high ability.

Assume that in each team $\alpha_1 > \alpha_2 = \alpha_3 = \alpha_4 = 1$. Then, the objective function becomes $a_1 \max\left((1+2(r_{13}-r_{11}))^{\frac{1}{\beta-1}},0\right) + \sum_{i=2}^{4} \max\left((1+2(r_{i3}-r_{i1}))^{\frac{1}{\beta-1}},0\right)$. The condition in Proposition 5 tells us that if all members exert positive effort then we must have:

$$\frac{1+2(r_{13}-r_{11})}{1+2(r_{i3}-r_{i1})} = \left(\frac{\alpha_1}{\alpha_i}\right)^{\frac{\beta-1}{\beta-2}}.$$

We also have the condition that $\sum_{i=i}^{4}(r_{i3}-r_{i1}) = 2$ as there are two more prizes to allocate when the team wins 3 rather than 1 prize. The other constraints are that $\Delta_i = (r_{i3}-r_{i1}) \leq 1$ as this is the maximum incentive that can be given, going from no prize to getting a prize for sure when the number of prizes won goes from 1 to 3. We can solve explicitly for the solution of this system of incentives. We have two unknowns Δ_1 and $\Delta_2 = \Delta_3 = \Delta_4$. We get

$$\begin{aligned}\Delta_1 &= \frac{7a^{\frac{\beta}{\beta-2}} - 3a^{\frac{1}{\beta-2}}}{2a^{\frac{\beta}{\beta-2}} + 6a^{\frac{1}{\beta-2}}}\\ \Delta_2 &= \frac{5a^{\frac{1}{\beta-2}} - a^{\frac{\beta}{\beta-2}}}{2a^{\frac{\beta}{\beta-2}} + 6a^{\frac{1}{\beta-2}}}\end{aligned}$$

Easy algebra shows that Δ_1 is decreasing in β. More convexity leads to more equal incentives. When β is close to 2, incentives should be unequal and the constraint $\Delta_1 \leq 1$ is binding. In that case, the optimal allocation rule is simply $\Delta_1 = 1$ and $\Delta_2 = \Delta_3 = \Delta_4 = 1/3$. Simple algebra shows that the constraint is always binding if $a \geq 1.8$.

Another possible rule selects one of the last three team members and gives him negative incentives, say $\Delta_4 = -1$ and give incentives $\Delta_1 = \Delta_2 = \Delta_3 = 1$ to the other 3 team members. We can compare total effort under the various cases. We consider the case with $a = 1.5$. For values of β larger than 4.223, the constraint $\Delta_1 \leq 1$ is not binding. Comparing the aggregate effort team effort under the $(\Delta_1 = 1,\ \Delta_2 = \Delta_3 = \Delta_4 = 1/3)$ and $(\Delta_1 = \Delta_2 = \Delta_3 = 1,\ \Delta_4 = -1)$ allocation rules, simple algebra shows that the latter dominates for values of $\beta \leq 2.449$.

The interpretation is simple. For values close to $\beta = 2$, unequal incentives are optimal. As the incentives given to the highest ability player are constrained, it it optimal to give highest incentives to three players.

For higher values of β, it becomes more efficient to give maximum incentives to the high ability player and give equal incentives to the remaining 3 players. When β is high, the high ability player does not receive the maximum incentives even if he gets higher incentives than the three other players.

5.5 Monotonic Allocation Rules

We may want to impose monotonicity as a desirable property of prize assignment rules. A rule described by an assignment matrix R_j is **monotonic** if and only if $r_{ij}^{k+1} \geq r_{ij}^k$ for all $j = 1, ..., n$ and $k = 1, ..., n$. Note that stochastic assignment rules are allowed in this definition. As is seen in Example 1, the optimal assignment matrix R_j does not necessarily satisfy monotonicity when $k^* > 1$. However, monotonicity is a reasonable requirement. An appealing property of monotonic rules is that every team members exerts positive effort. Rewriting $\sum_{k=1}^n r_{ij}^k \mu^k(p_j^*)$:

$$\sum_{k=1}^n r_{ij}^k \mu^k(p_j^*) = r_j^1 \sum_{k=1}^n \mu^k(p_j^*) + \left(r_j^2 - r_j^1\right) \sum_{k=2}^n \mu^k(p_j^*) + ... + \left(r_j^n - r_j^{n-1}\right) \mu^n(p_j^*)$$

By the first order stochastic dominance, $\sum_{k=m}^n \mu^k(p_j^*) > 0$ for all $m = 1, ..., n$. By monotonicity, $r_{ij}^k - r_{ij}^{k-1} \geq 0$ for all $k = 1, ..., n$ ($r_0^k = 0$). Thus, monotonicity implies:

$$\max\left(\sum_{k=1}^n r_{ij}^k \mu^k(p_j^*), 0\right) = \sum_{k=1}^n r_{ij}^k \mu^k(p_j^*) > 0$$

Proposition 6. Under any monotonic rule, every team member exerts effort.

Under deterministic rules, monotonicity requires that if team member j wins a prize when m prizes are won, she will also win a prize if more than m prizes are won. Then, the incentive corresponding to the mth position on the list can be simplified to:

$$M_m(p_j^*) = \sum_{k=m}^n \mu^k(p_j^*)$$

Order $M_m(p_j^*)$s by their values, and define a one-to-one mapping $m^* : \{1, ..., i, ..., n\} \to \{1, ..., k, ..., n\}$ such that

$$M_{m^*(1)}(p_j^*) \geq .... \geq M_{m^*(j)}(p_j^*) \geq ... \geq M_{m^*(n)}(p_j^*)$$

The following result is then straightforward.

Proposition 7. The list rule $m^* : \{1, ..., n\} \to \{1, ..., n\}$ is the optimal monotonic assignment rule.

Note that $m^*(1) = k^*$. Thus, the assignment of the most able team member is the same in the optimal deterministic rule as in the optimal list rule. Under the single-crossing condition, it is easy to see that $m^*(2)$ is either $k^* + 1$ or $k^* - 1$, and m^* orders team members in such a way that it forms a single-peaked way at peak $m^*(1) = k^*$.[11]

These findings generate implications for our main application, elections under closed-list proportional representation. Proposition 7 implies that, when parties decide how to rank their candidates to maximize effort incentives, candidates should not be ranked in decreasing order of effectiveness. Top spots are safe seats. Candidates in these positions are sure to get a seat and have little incentive to exert effort. Candidates in the middle of the list close to the number of seats that the party is expected to win are in hot seats and have maximal incentives to exert effort.

Yet, real world electoral lists often rank candidates in decreasing in order of effectiveness or competence; see for example Galasso and Nannicini (2015), Dal Bo et al. (2017), Cirone, Cox and Fiva (2020), Fujiwara and Sanz (2020), Cox et al. (2021) and Buisseret et al. (2022). This empirical literature also shows that party leaders and other top brass party members are typically the most competent candidates parties put forward in elections and are located at the top of the list. Our theory focuses on only one source of incentives, the prospect of winning a legislative seat. Yet, parties face a menu of incentive devices and constraints. For example, party leaders and other top brass party members are incentivized by media coverage (which typically focuses on them) and by the prospect of having access to additional, post-electoral offices or benefits (on the effects of these incentive levers, see Cox et al. 2021 and Crutzen, Konishi and Sahuguet 2022).

Lists lead by construction to monotonic allocations in which the probability that a candidate wins a seat increases with the numbers of seats won by her party. Our findings in section 5.3 and 5.4 imply that the optimal allocation rule is usually non-monotonic (see for example the last rows of the optimal assignment

[11] An interesting special case is $k^* = 1$. Since $\mu^k(p_j^*) > 0$ for all $k = 1, ..., n$ by single-crossing, we have $m^*(i) = i$ for all $i = 1, ...n$, and that list rule is the optimal prize assignment rule.

matrices in example 1 of section 5.3). One way to reconcile these findings with real-world incentives in closed-list proportional representation systems is to consider other sources of incentives. Top party candidates usually keep their (control of the party and) their legislative seat only if the party does well in the election; see Pilet and Cross (2014) and O'Brien (2015). This means that when the party does not perform well in the election, these top candidates are replaced. This generates non-monotonic incentives for these replacement candidates.

6 Concluding Remarks

In this paper, we analyzed a model of team contest for a set of indivisible homogeneous prizes. The number of prizes won by a team depends on the aggregate effort of its members, and each team member can be awarded at most one prize. Team members exert effort as a function of the prize allocation rule of their team. We derive the unique Nash equilibrium of effort choices for any profile of allocation rules. We then prove that the uniqueness of the perfect Bayesian equilibrium in the two-stage game in which teams choose allocation rules before team members exert effort. We then characterize the optimal allocation rule as a function of the abilities of team members. We show that the allocation rule depends in an important way on the degree of convexity of the cost function. When the cost is not too convex, the allocation rule maximizes the difference of incentives and gives the strongest incentives to the highest ability players. When the cost function is very convex, the allocation rules gives incentives to team members in a way that is proportional to their abilities. In both cases, we show that the optimal rule is in general not monotonic in the sense that some players can get a prize with a lower probability when their team wins more prizes.

Appendix

Proof of Lemma 1. Without loss of generality, we assume $n_j = n$. To simplify notation, we drop team j subscripts. We will prove the statement by induction. Let's start with $n = 3$. In this case, it is easy to see (i) if $k = 1$, then we can set $q^1(\{i\}) = r_i^1$ for all $i = 1, 2, 3$, (ii) if $k = 2$, we can set $q^2(N\backslash\{i\}) = 1 - r_i^2$ for all $i = 1, 2, 3$, and (iii) $q^3(\{1, 2, 3\}) = 1$ since $r_i^3 = 1$ must hold for all $i = 1, 2, 3$. Thus, for $n = 3$ we can find $(q^k)_{k=1}^3$ for any feasible R.

Now, suppose that for $n = m$ we can find $(q^k)_{k=1}^m$ for any $m \times m$ matrix R with $r_i^k \in [0, 1]$ and $\sum_{i=1}^m r_i^k = k$ for all $k = 1, ..., m$ and $i = 1, ..., m$. We will show that for $n = m + 1$ we can find $(q^k)_{k=1}^{m+1}$ for any $(m+1) \times (m+1)$ matrix R with $r_i^k \in [0, 1]$ and $\sum_{i=1}^{m+1} r_i^k = k$ for all $k = 1, ..., m+1$ and $i = 1, ..., m+1$.

Let $n = m + 1$. As in the case of $n = 3$, we can see that for $k = 1$, 2, and $m + 1$, we can find q^ks. We will show for all other $k = 2, ..., m$, we can find $q^k : \mathcal{S}(k, N_i \cup \{m + 1\}) \to [0, 1]$ with $N_i = \{1, ..., m\}$ for all $(r_1^k, ..., r_{m+1}^k)$ with $\sum_{i=1}^{m+1} r_i^k = k$. Let $i^* \in \arg\max_i r_i^k$, and let $r_{-i^*}^k = (r_1^k, ..., r_{i^*-1}^k, r_{i^*+1}^k, ..., r_{m+1}^k)$.

First, let $\bar{r}^k = r_{-i^*}^k \times \frac{k}{|r_{-i^*}^k|}$. Since $\bar{r}^k$ has m arguments, we can find $\bar{q}^k : \mathcal{S}(k, N_i) \to [0, 1]$ with $|N_i| = m$ which supports $\bar{r}^k$ by our induction hypothesis. Then, we can create $\hat{q}^k : \mathcal{S}(k, N_i \cup \{i^*\}) \to [0, 1]$ which supports $\hat{r}^k = (\underbrace{\bar{r}^k}_{-i^*}, \underbrace{0}_{i^*})$ by setting $\hat{q}^k(S) = \bar{q}^k(S)$ for all $S \in (k, N_i)$ with $\bar{q}^k(S) > 0$, and $\hat{q}^k(S) = 0$ for any other $S \in \mathcal{S}(k, N_i \cup \{i^*\})$.

Second, let $\bar{r}^{k-1} = r_{-i^*}^k \times \frac{k-1}{|r_{-i^*}^k|}$. Since $\bar{r}^{k-1}$ has m arguments and $k \geq 2$, we can find $\bar{q}^{k-1} : \mathcal{S}(k - 1, N_i) \to [0, 1]$ with $|N_i| = m$ which supports $\bar{r}^{k-1}$ by our induction hypothesis. Then, we can create $\check{q}^k : \mathcal{S}(k, N_i \cup \{i^*\}) \to [0, 1]$ which supports $\check{r}^k = (\underbrace{\bar{r}^{k-1}}_{-i^*}, \underbrace{1}_{i^*})$ by setting $\check{q}^k(S \cup \{i^*\}) = \bar{q}^{k-1}(S)$ for all $S \in (k, N_i)$ with $\bar{q}^{k-1}(S) > 0$, and $\hat{q}^k(S) = 0$ for any other $S \in \mathcal{S}(k, N_i \cup \{i^*\})$.

Clearly, $r^k = (r_{-i^*}^k, r_{i^*}^k)$ can be written as a convex combination of $\hat{r}^k$ and $\check{r}^k$. This implies that r^k can be supported by a convex combination of $\hat{q}^k$ and $\check{q}^k$. Thus, we proved the induction hypothesis for $n = m + 1$.

This completes the proof.■

We now check whether the solution generated from the first-order conditions satisfy the second-order conditions. The following lemma provides a sufficient condition.

Proof of Lemma 2. From the first-order conditions (3), we have:

$$\frac{\partial B_{ij}}{\partial e_{ij}} - c'_{ij}(e_{ij}) = \frac{\gamma V}{E_j} a_{ij} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} - c_{ij} e_{ij}^{\beta-1} = 0.$$

$$\sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) = \sum_{k=1}^{n_j} r_i^k C(n,k) p_j^k (1-p_j)^{n-k}(k-np_j) + (n-n_j) C(n,n_j) p_j^{n-n_j} (1-p_j)^{n_j+1} \geq 0$$

To analyze the second-order conditions, we differentiate $\mu^k(p_j) = C(n,k)p_j^k (1-p_j)^{n-k}(k-np_j)$ and $\nu^{n_j}(p_j) = (n-n_j)\, C(n,n_j) p_j^{n-n_j}(1-p_j)^{n_j+1}$:

$$\begin{aligned}
\frac{d\mu^k(p_j)}{dp_j} &= C(n,k)p_j^{k-1}(1-p_j)^{n-k-1}\left[k(1-p_j)(k-np_j) - (n-k)\, p_j (k-np_j) - np_j(1-p_j)\right] \\
&= C(n,k)p_j^{k-1}(1-p_j)^{n-k-1}\left[(k-np_j)^2 - np_j(1-p_j)\right] \\
&= C(n,k)p_j^{k}(1-p_j)^{n-k}(k-np_j)\left[\frac{k-np_j}{p_j(1-p_j)} - \frac{n}{k-np_j}\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\nu^{n_j}(p_j)}{dp_j} &= (n-n_j)\, C(n,n_j) p_j^{n-n_j-1}(1-p_j)^{n_j}\left[(n-n_j)(1-p_j) - n_j p_j\right] \\
&= (n-n_j)\, C(n,n_j) p_j^{n-n_j-1}(1-p_j)^{n_j}\left[(n-n_j) - np_j\right] \\
&= (n-n_j)\, C(n,n_j) p_j^{n-n_j}(1-p_j)^{n_j+1}\frac{\left[(n-n_j) - np_j\right]}{p_j(1-p_j)}
\end{aligned}$$

Thus, the second-order derivative becomes:

$$
\begin{aligned}
& \frac{\partial^2 B_{ij}}{\partial e_{ij}^2} - c^{''}(e_{ij}) \\
= \quad & -\gamma V \frac{a_{ij}^2}{E_j^2} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} + \gamma^2 V \frac{a_{ij}^2}{E_j^2} p_j (1-p_j) \left\{ \sum_{k=1}^{n_j} r_i^k \frac{d\mu^k(p_j)}{dp_j} + \frac{d\nu^{n_j}(p_j)}{dp_j} \right\} - (\beta - 1) c_{ij} e_{ij}^{\beta-2} \\
= \quad & -\gamma V \frac{a_{ij}^2}{E_j^2} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} + \gamma^2 V \frac{a_{ij}^2}{E_j^2} p_j (1-p_j) \left\{ \sum_{k=1}^{n_j} r_i^k \frac{d\mu^k(p_j)}{dp_j} + \frac{d\nu^{n_j}(p_j)}{dp_j} \right\} \\
& -\frac{(\beta-1)}{e_{ij}} \frac{\gamma V a_{ij}}{E_j} \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} \\
= \quad & -\gamma V \frac{a_{ij}^2}{E_j^2} \left(1 + \frac{(\beta-1) E_j}{a_{ij} e_{ij}} \right) \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} + \gamma V \frac{a_{ij}^2}{E_j^2} \times \gamma p_j (1-p_j) \left\{ \sum_{k=1}^{n_j} r_i^k \frac{d\mu^k(p_j)}{dp_j} + \frac{d\nu^{n_j}(p_j)}{dp_j} \right\} \\
= \quad & \gamma V \frac{a_{ij}^2}{E_j^2} \left[-\left(1 + \frac{(\beta-1) E_j}{a_{ij} e_{ij}} \right) \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) + \nu^{n_j}(p_j) \right\} \right. \\
& \left. \quad + \gamma \left\{ \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) \left(k - np_j - \frac{np_j(1-p_j)}{k - np_j} \right) + \nu^{n_j}(p_j) \left[(n - n_j) - np_j \right] \right\} \right] \\
= \quad & \gamma V \frac{a_{ij}^2}{E_j^2} \left[\sum_{k=1}^{n_j} r_i^k \mu^k(p_j) \left\{ \gamma \left(k - np_j - \frac{np_j(1-p_j)}{k-np_j} \right) - \left(1 + \frac{(\beta-1)}{\theta_{ij}} \right) \right\} \right. \\
& \left. \quad + \nu^{n_j}(p_j) \left\{ \gamma \left((n-n_j) - np_j \right) - \left(1 + \frac{(\beta-1)}{\theta_{ij}} \right) \right\} \right] \\
\leq \quad & 0.
\end{aligned}
$$

where $\theta_{ij} \equiv \frac{a_{ij}e_{ij}}{E_j}$ is member i's effective effort share. Thus, $\frac{\partial^2 B_{ij}}{\partial e_{ij}^2} - c^{''}(e_{ij}) < 0$ if and only if

$$
\begin{aligned}
& \sum_{k=1}^{n_j} r_i^k \mu^k(p_j) \left\{ \gamma \left(k - np_j - \frac{np_j(1-p_j)}{k-np_j} \right) - \left(1 + \frac{(\beta-1)}{\theta_{ij}} \right) \right\} \\
& + \nu^k(p_j) \left\{ \gamma \left((n-n_j) - np_j \right) - \left(1 + \frac{(\beta-1)}{\theta_{ij}} \right) \right\} < 0.
\end{aligned}
$$

Ths implies that if (a) $\gamma \left(k - np_j - \frac{np_j(1-p_j)}{k-np_j} \right) - \left(1 + \frac{(\beta-1)}{\theta_{ij}} \right) \leq 0$ for each $np_j < k \leq n_j$, and (b) $\gamma \left((n-n_j) - np_j \right) - \left(1 + \frac{(\beta-1)}{\theta_{ij}} \right) \leq 0$, the second-order conditions are satisfied. For (a), since $k = n_j$ is the hardest to satisfy, if $\gamma \leq \frac{1 + \frac{(\beta-1)}{\theta_{ij}}}{n_j - np_j}$ is satisfied, it is sufficient for the contents of the first brace to be negative, since

$$
\gamma \leq \frac{1 + \frac{(\beta-1)}{\theta_{ij}}}{n_j - np_j} < \frac{1 + \frac{(\beta-1)}{\theta_{ij}}}{n_j - np_j - \frac{np_j(1-p_j)}{n_j - np_j}}.
$$

Similarly, for the contents of the second brace to be negative, we need $\gamma \leq \frac{1 + \frac{(\beta-1)}{\theta_{ij}}}{(n-n_j) - np_j}$. We conclude that the second-order conditions are satisfied if $\gamma \leq \frac{1 + \frac{(\beta-1)}{\theta_{ij}}}{\max\{n_j - np_j, n - n_j - np_j\}}$ holds for all $i = 1, ..., n_j$.$\square$

Proof of Proposition 2.

An equilibrium is described by the following system of equations:[12]

$$\begin{pmatrix} p_1 \\ \vdots \\ p_j \\ \vdots \\ p_J \end{pmatrix} = \begin{pmatrix} \frac{E_1^\gamma(p_1)}{\sum_{\ell=1}^J E_\ell^\gamma(p_\ell)} \\ \vdots \\ \frac{E_j^\gamma(p_j)}{\sum_{\ell=1}^J E_\ell^\gamma(p_\ell)} \\ \vdots \\ \frac{E_J^\gamma(p_J)}{\sum_{\ell=1}^J E_\ell^\gamma(p_\ell)} \end{pmatrix}$$

First, we drop the last equation by setting $p_J = 1 - \sum_{j=1}^{J-1} p_j$, since $\sum_{j=1}^{J} p_j = 1$. Let $E \equiv \sum_{\ell=1}^{J-1} E_\ell^\gamma(p_\ell) + E_J^\gamma(1 - \sum_{j=1}^{J-1} p_j)$. We totally differentiate the system with $(p_1, ..., p_{J-1})$ to obtain:

$$\begin{pmatrix} dp_1 \\ \vdots \\ dp_{J-1} \end{pmatrix} = \begin{pmatrix} \frac{\gamma \frac{E_1^\gamma}{E_1} \frac{\partial E_1}{\partial p_1}}{E} - \frac{\gamma \frac{E_1^\gamma}{E_1} E_1^\gamma \frac{\partial E_1}{\partial p_1}}{E^2} - \frac{\gamma \frac{E_J^\gamma}{E_J} E_1^\gamma \frac{\partial E_J}{\partial p_1}}{E^2} & \cdots & -\frac{\gamma \frac{E_{J-1}^\gamma}{E_{J-1}} E_1^\gamma \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} - \frac{\gamma \frac{E_J^\gamma}{E_J} E_1^\gamma \frac{\partial E_J}{\partial p_{J-1}}}{E^2} \\ \vdots & \ddots & \vdots \\ -\frac{\gamma \frac{E_1^\gamma}{E_1} E_{J-1}^\gamma \frac{\partial E_1}{\partial p_1}}{E^2} - \frac{\gamma \frac{E_J^\gamma}{E_J} E_{J-1}^\gamma \frac{\partial E_J}{\partial p_1}}{E^2} & \cdots & \frac{\gamma \frac{E_{J-1}^\gamma}{E_{J-1}} \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E} - \frac{\gamma \frac{E_{J-1}^\gamma}{E_{J-1}} E_{J-1}^\gamma \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} - \frac{\gamma \frac{E_J^\gamma}{E_J} E_{J-1}^\gamma \frac{\partial E_J}{\partial p_{J-1}}}{E^2} \end{pmatrix} \begin{pmatrix} dp_1 \\ \vdots \\ dp_{J-1} \end{pmatrix}$$

Denote the above $(J-1) \times (J-1)$ matrix by D.

We first show that the determinant of D can be written as:

$$|D| = \prod_{j=1}^{J} (1 - \gamma\eta_j) \left(\sum_{k=1}^{J} \frac{p_k}{1 - \gamma\eta_k} \right),$$

where $\eta_j = \frac{p_j \partial E_j}{E_j \partial p_j}$ is party j's winning probability elasticity of total effort.

First note that $\frac{\partial E_J(1 - \Sigma_{k=1}^{J-1} p_k)}{\partial p_j} = -\frac{\partial E_J(p_J)}{\partial p_J}$. This implies

$$\frac{\gamma \frac{E_J^\gamma}{E_J} E_j^\gamma \frac{\partial E_J}{\partial p_j}}{E^2} = -\frac{\gamma \frac{E_J^\gamma}{E_J} E_j^\gamma \frac{\partial E_J}{\partial p_J}}{E^2} = -\gamma p_j \frac{p_J}{E_J} \frac{\partial E_J}{\partial p_J}.$$

[12] This analysis is valid for any prize allocation rule and for any functional form of the effort aggregator function.

Let the matrix in the left-hand side be D, and let $\eta_j = \frac{p_j}{E_j}\frac{\partial E_j}{\partial p_j}$. Then, the determinant of D is

$$
\begin{aligned}
& |D| \\
= & \begin{vmatrix}
1-\gamma\eta_1+\gamma p_1\eta_1-\gamma p_1\eta_J & \cdots & \gamma p_1\eta_j-\gamma p_1\eta_J & \cdots & \gamma p_1\eta_{J-1}-\gamma p_1\eta_J \\
\vdots & \ddots & \vdots & & \vdots \\
\gamma p_j\eta_1-\gamma p_j\eta_J & \cdots & 1-\gamma\eta_j+\gamma p_j\eta_j-\gamma p_j\eta_J & \cdots & \gamma p_j\eta_{J-1}-\gamma p_j\eta_J \\
\vdots & & \vdots & \ddots & \vdots \\
\gamma p_{J-1}\eta_1-\gamma p_{J-1}\eta_J & \cdots & \gamma p_{J-1}\eta_j-\gamma p_{J-1}\eta_J & \cdots & 1-\gamma\eta_{J-1}+\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta_J
\end{vmatrix} \\
= & \begin{vmatrix}
1-\gamma\eta_1 & & 0 & & 0 & -\frac{p_1}{p_{J-1}}(1-\gamma\eta_{J-1}) \\
\vdots & \ddots & & & \vdots & \vdots \\
0 & & 1-\gamma\eta_j & & 0 & -\frac{p_j}{p_{J-1}}(1-\gamma\eta_{J-1}) \\
\vdots & & & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 1-\gamma\eta_{J-2} & -\frac{p_{J-2}}{p_{J-1}}(1-\gamma\eta_{J-1}) \\
\gamma p_{J-1}\eta_1-\gamma p_{J-1}\eta_J & & \gamma p_{J-1}\eta_j-\gamma p_{J-1}\eta_J & & \gamma p_{J-1}\eta_{J-2}-\gamma p_{J-1}\eta_J & 1-\gamma\eta_{J-1}+\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta_J
\end{vmatrix} \\
= & \prod_{j=1}^{J-2}(1-\gamma\eta_j)(1-\gamma\eta_{J-1}+\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta_J)+\prod_{j=1}^{J-1}(1-\gamma\eta_j)\sum_{k=1}^{J-2}\frac{\gamma p_k\eta_k-\gamma p_k\eta_J}{1-\gamma\eta_k} \\
= & \prod_{j=1}^{J-1}(1-\gamma\eta_j)+\prod_{j=1}^{J-1}(1-\gamma\eta_j)\frac{(\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta_J)}{1-\gamma\eta_{J-1}}+\prod_{j=1}^{J-1}(1-\gamma\eta_j)\sum_{k=1}^{J-2}\frac{\gamma p_k\eta_k-\gamma p_k\eta_J}{1-\gamma\eta_k} \\
= & \prod_{j=1}^{J-1}(1-\gamma\eta_j)+\prod_{j=1}^{J-1}(1-\gamma\eta_j)\sum_{k=1}^{J-1}p_k\frac{\gamma\eta_k-\gamma\eta_J}{1-\gamma\eta_k} \\
= & \prod_{j=1}^{J}(1-\gamma\eta_j)\left(\frac{1}{1-\gamma\eta_J}+\frac{1}{1-\gamma\eta_J}\sum_{k=1}^{J-1}p_k\frac{(1-\gamma\eta_J)-(1-\gamma\eta_k)}{1-\gamma\eta_k}\right) \\
= & \prod_{j=1}^{J}(1-\gamma\eta_j)\left(\frac{1}{1-\gamma\eta_J}-\frac{\sum_{k=1}^{J-1}p_k}{1-\gamma\eta_J}+\sum_{k=1}^{J-1}p_k\frac{1}{1-\gamma\eta_k}\right) \\
= & \prod_{j=1}^{J}(1-\gamma\eta_j)\left(\sum_{k=1}^{J}\frac{p_k}{1-\gamma\eta_k}\right)
\end{aligned}
$$

Thus, $|D|>0$ holds if $\gamma\eta_j<1$ is satisfied for all $j=1,...J$.

We now show that $\gamma\eta_j<1$ for any assignment matrix R_j if $\gamma \leq \frac{\beta}{\max\{n_j, n-n_j\}}$.

Recalling $E_j(p_j, R_j) = (\gamma V)^{\frac{1}{\beta(\beta-1)}} \left[\sum_{h=1}^{n_j} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}} \right]^{\frac{1}{\beta}}$, we have

$$
\begin{aligned}
\eta_j &= \frac{1}{\beta} p_j \left[\sum_{h=1}^{n_j} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}} \right]^{\frac{-1}{\beta}} \left[\sum_{h=1}^{n} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}} \right]^{\frac{1}{\beta}-1} \left[\sum_{h=1}^{n_j} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}-1} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho'_{ij}(p_j) \right) \right] \\
&= \frac{\sum_{h=1}^{n_j} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}-1} \left(\frac{a_{ij}^{\beta}}{c_{ij}} p_j \rho'_{ij}(p_j) \right)}{\beta \sum_{h=1}^{n_j} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}}} \\
&= \frac{\sum_{h=1}^{n_j} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}} \frac{p_j \rho'_{ij}(p_j)}{\rho_{ij}(p_j)}}{\beta \sum_{h=1}^{n_j} \left(\frac{a_{ij}^{\beta}}{c_{ij}} \rho_{ij}(p_j) \right)^{\frac{1}{\beta-1}}} \\
&\leq \frac{1}{\beta} \max_{h=1,\ldots,n_j} \left\{ \frac{p_j \rho'_{hj}(p_j)}{\rho_{hj}(p_j)} \right\}
\end{aligned}
$$

Since $\rho_{ij}(p_j) = \max\left\{ \sum_{k=1}^{n_j} r_{ij}^k \mu^k(p_j) + \nu^{n_j}(p_j), 0 \right\}$, where $\mu^k(p_j) = C(n,k) p^k (1-p_j)^{n-k}(k - np_j)$ and $\nu^{n_j}(p_j) = (n - n_j)\, C(n, n_j) p_j^{n-n_j} (1-p_j)^{n_j+1}$, $\frac{p_j \rho'_{hj}(p_j)}{\rho_{hj}(p_j)}$ is bounded above by $\max\limits_{k \geq np_j} \left\{ \frac{p_j \partial \mu^k(p_j)/\partial p_j}{\mu^k(p_j)} \right\}$ and $\frac{p_j \partial \nu^k(p_j)/\partial p_j}{\nu^k(p_j)}$. Direct calculations show:

$$
\begin{aligned}
\frac{p_j \partial \mu^k(p_j)/\partial p_j}{\mu^k(p_j)} &= \frac{(k - np_j)^2 - np_j\,(1-p_j)}{(1-p_j)\,(k - np_j)} \\
&= \frac{k - np_j}{1 - p_j} - \frac{np_j}{k - np_j} \\
&\leq \frac{n_j - np_j}{1 - p_j} - \frac{np_j}{n_j - np_j} < \frac{n_j - np_j}{1 - p_j} < n_j,
\end{aligned}
$$

and

$$
\frac{p_j \partial \nu^k(p_j)/\partial p_j}{\nu^k(p_j)} = \frac{(n - n_j) - np_j}{(1 - p_j)} < n - n_j
$$

Thus, we conclude $\eta_j < \frac{\max\{n_j, n-n_j\}}{\beta}$, and $\gamma \eta_j < 1$ becomes $\gamma < \frac{\beta}{\max\{n_j, n-n_j\}}$.

We then use the index theorem (Varian 1975 or Mas-Colell et al (1995) to prove uniqueness of the equilibrium. Continuity of equilibrium function $p^*(R)$ follows from the implicit function theorem.$\square$

Proof of Proposition 4.

Totally differentiating the system including Δ_j, we obtain:

$$\begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} = \begin{pmatrix} & & \\ & D & \\ & & \end{pmatrix} \begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} + \begin{pmatrix} -\frac{\gamma \frac{E_j^\gamma}{E_j} E_1^\gamma \frac{\partial E_j}{\partial \Delta}}{E^2} \\ \vdots \\ \frac{\gamma \frac{E_j^\gamma}{E_j} \frac{\partial E_j}{\partial \Delta}}{E} - \frac{\gamma \frac{E_j^\gamma}{E_j} E_j^\gamma \frac{\partial E_j}{\partial \Delta}}{E^2} \\ \vdots \\ -\frac{\gamma \frac{E_j^\gamma}{E_j} E_{J-1}^\gamma \frac{\partial E_j}{\partial \Delta}}{E^2} \end{pmatrix} \left(\frac{a_{ij}^\beta}{c_{i1}} - \frac{a_{hj}^\beta}{c_{h1}} \right) \mu^k(p_j) d\Delta_j$$

Rewriting the above by using $\eta_j = \frac{p_j}{E_j}\frac{\partial E_j}{\partial p_j}$, $\varphi_j = \frac{p_j}{E_j}\frac{\partial E_j}{\partial \Delta_j}$ and $p_k = \frac{E_k^\gamma}{E}$, we have:

$$\begin{pmatrix} 1-\gamma\eta_1+\gamma p_1\eta_1-\gamma p_1\eta_J & \cdots & \gamma p_1\eta_j-\gamma p_1\eta_J & \cdots & \gamma p_1\eta_{J-1}-\gamma p_1\eta_J \\ \vdots & \ddots & \vdots & & \vdots \\ \gamma p_j\eta_1-\gamma p_j\eta_J & \cdots & 1-\gamma\eta_j+\gamma p_j\eta_j-\gamma p_j\eta_J & \cdots & \gamma p_j\eta_{J-1}-\gamma p_j\eta_J \\ \vdots & & \vdots & \ddots & \vdots \\ \gamma p_{J-1}\eta_1-\gamma p_{J-1}\eta_J & \cdots & \gamma p_{J-1}\eta_j-\gamma p_{J-1}\eta_J & \cdots & 1-\gamma\eta_{J-1}+\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta_J \end{pmatrix} \begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix}$$

$$= \gamma\varphi_j \left(\frac{a_{ij}^\beta}{c_{i1}} - \frac{a_{hj}^\beta}{c_{h1}} \right) \mu^k(p_j) \begin{pmatrix} -p_1 \\ \vdots \\ 1-p_j \\ \vdots \\ -p_{J-1} \end{pmatrix} d\Delta_j$$

We can prove following result

$$\frac{dp_j}{d\Delta_j} = \frac{\gamma\varphi_j}{|D|} \left(\frac{a_{ij}^\beta}{c_{i1}} - \frac{a_{hj}^\beta}{c_{h1}} \right) \mu^k(p_j) \left[\prod_{k=1,k\neq j}^{J} (1-\gamma\eta_k) \left\{ \sum_{k=1,k\neq j}^{J} \frac{p_k}{1-\gamma\eta_k} \right\} \right].$$

Direct calculations yield:

$$\frac{dp_j}{d\Delta_j} = \frac{\gamma\varphi_j}{|D|}\left(\frac{a_{ij}^{\beta}}{c_{i1}} - \frac{a_{hj}^{\beta}}{c_{h1}}\right)\mu^k(p_j)\begin{vmatrix} 1-\gamma\eta_1+\gamma p_1\eta_1-\gamma p_1\eta_J & \cdots & -p_1 & \cdots & \gamma p_1\eta_{J-1}-\gamma p_1\eta_J \\ \vdots & \ddots & \vdots & & \vdots \\ \gamma p_j\eta_1-\gamma p_j\eta_J & \cdots & 1-p_j & \cdots & \gamma p_j\eta_{J-1}-\gamma p_j\eta_J \\ \vdots & & \vdots & \ddots & \vdots \\ \gamma p_{J-1}\eta_1-\gamma p_{J-1}\eta_J & \cdots & -p_{J-1} & \cdots & 1-\gamma\eta_{J-1}+\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta_J \end{vmatrix}$$

$$= \frac{\gamma\varphi_j}{|D|}\left(\frac{a_{ij}^{\beta}}{c_{i1}} - \frac{a_{hj}^{\beta}}{c_{h1}}\right)\mu^k(p_j)$$

$$\times\begin{vmatrix} 1-\gamma\eta_1 & 0 & 0 & -\frac{p_1}{p_{J-1}}(1-\gamma\eta_{J-1}) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & -\frac{p_j}{p_{J-1}}(1-\gamma\eta_{J-1}) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots\ 0\ \cdots & 1-\gamma\eta_{J-2} & -\frac{p_{J-2}}{p_{J-1}}(1-\gamma\eta_{J-1}) \\ \gamma p_{J-1}\eta_1-\gamma p_{J-1}\eta_J & -p_{J-1} & \gamma p_{J-1}\eta_{J-2}-\gamma p_{J-1}\eta_J & 1-\gamma\eta_{J-1}+\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta_J \end{vmatrix}$$

$$= \frac{\gamma\varphi_j}{|D|}\left(\frac{a_{ij}^{\beta}}{c_{i1}} - \frac{a_{hj}^{\beta}}{c_{h1}}\right)\mu^k(p_j)$$

$$\times\left[\prod_{k=1,k\neq j}^{J-2}(1-\gamma\eta_k)(1-\gamma\eta_{J-1}+\gamma p_{J-1}\eta_{J-1}-\gamma p_{J-1}\eta) + \prod_{k=1,k\neq j}^{J-1}(1-\gamma\eta_k)\sum_{k=1,k\neq j}^{J-2}\frac{\gamma p_k\eta_k-\gamma p_k\eta_J}{1-\gamma\eta_k}\right]$$

$$= \frac{\gamma\varphi_j}{|D|}\left(\frac{a_{ij}^{\beta}}{c_{i1}} - \frac{a_{hj}^{\beta}}{c_{h1}}\right)\mu^k(p_j)\left[\prod_{k=1,k\neq j}^{J-1}(1-\gamma\eta_k) + \prod_{k=1,k\neq j}^{J-1}(1-\gamma\eta_k)\sum_{k=1,k\neq j}^{J-1}\frac{\gamma p_k\eta_k-\gamma p_k\eta_J}{1-\gamma\eta_k}\right]$$

$$= \frac{\gamma\varphi_j}{|D|}\left(\frac{a_{ij}^{\beta}}{c_{i1}} - \frac{a_{hj}^{\beta}}{c_{h1}}\right)\mu^k(p_j)\left[\prod_{k=1,k\neq j}^{J}(1-\gamma\eta_k)\left\{\sum_{k=1,k\neq j}^{J}\frac{p_k}{1-\gamma\eta_k}\right\}\right].$$

This result shows that when $\frac{a_{ij}^{\beta}}{c_{i1}} > \frac{a_{hj}^{\beta}}{c_{h1}}$, $\frac{dp_j}{d\Delta_j} > 0$ holds. It also implies that each party j can increase p_j by adjusting assignment rule R to increase its aggregate effort $\sum_{h=1}^{n}\frac{a_{hj}^{\beta}}{c_{hj}}\max\left\{\sum_{k=1}^{n} r_{hj}^k\mu^k(p_j), 0\right\}$.$\square$

Proof of Theorem 1. Using Proposition 4, let $\tilde{\xi}_j : R^J \times \Delta^J \twoheadrightarrow R$ be such that $\tilde{\xi}_j(R_1, ..., R_J, p_1, ..., p_J) = \arg\max_{R_j\in\mathcal{R}} E_j(p_j, R_j)$. This is nonempty-valued and upper hemicontinuous by Weierstrass's theorem and Berge's maximum theorem. As will be seen, π_j is concave or convex in R_j depending on $\beta > 2$ or $\beta \leq 2$.

If π_j is concave, $\xi_j(p_j) \equiv \tilde{\xi}_j(p_j)$ is convex-valued. If π_j is convex, the best response is a deterministic: $\tilde{\xi}_j(p_j) \subset R^D \equiv \{R_j \in \mathcal{R} : r_i^k \in \{0,1\}\}$. Let $\xi_j(p_j) \equiv \left\{R_j \in \mathcal{R} : R_j \text{ is a convex combination of } \tilde{\xi}_j(p_j)\right\}$, which is convex-valued as well. Thus, independent of the value of β, ξ_j is nonempty-valued, upper hemicontinuous, and convex-valued. Let $\xi : R^J \times \Delta^J \twoheadrightarrow R^J$ be the Cartesian product of ξ_js.

Let $\psi : \Delta^J \times R^J \to \Delta^J$ be such that $\psi(p_1^*, ..., p_J^*, R_1^*, ..., R_J^*) = \left(\frac{E_1^\gamma(p_1,R_1)}{\sum_{j=1}^J E_1^\gamma(p_j,R_j)}, ..., \frac{E_J^\gamma(p_1,R_1)}{\sum_{j=1}^J E_1^\gamma(p_j,R_j)} \right)$. This is a continuous function by Proposition 3. Let $\phi : \Delta^J \times R^J \twoheadrightarrow \Delta^J \times R^J$ be a product of $\psi : \Delta^J \times R^J \to \Delta^J$ and $\xi : \Delta^J \times R^J \twoheadrightarrow R^J$. This is a fixed point mapping that satisfies all conditions of Kakutani's fixed point theorem. Thus, we have $(p_1^*, ..., p_J^*, R_1^*, ..., R_J^*) \in \phi(p_1^*, ..., p_J^*, R_1^*, ..., R_J^*)$. That is, a pair $(p_1^*, ..., p_J^*, R_1^*, ..., R_J^*)$ is the equilibrium path of a subgame perfect equilibrium with $p^*(R)$ in Proposition 3 for all possible subgames $R \in R^J$. By construction, it is easy to see $p^* \equiv (p_1^*, ..., p_J^*) = p^*(R^*)$.□

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