HOW (NOT) TO REFORM INDIA'S AFFIRMATIVE ACTION POLICIES FOR ITS ECONOMICALLY WEAKER SEGMENTS

TAYFUN SÖNMEZ AND M. UTKU ÜNVER

ABSTRACT. Various groups in India are protected with a vertical reservation (VR) policy, which sets aside a fraction of government positions and public school seats for each protected group. By law, VR-protected positions are processed after those open to all applicants, thus assuring that they are awarded to individuals who cannot receive open positions with merit. Historically, VR protections were exclusive to groups who suffered marginalization and discrimination due to their hereditary caste identities. This structure means that no individual can belong to multiple VR-protected groups, which in turn implies that the processing sequence of VR-protected groups is immaterial. A Constitutional Amendment in 2019 granted economically weaker sections (EWS) with VR protections, but limited its eligibility to members of forward castes who are ineligible for caste-based VR protections. The amendment was immediately brought to court, and the exclusion of members of caste-based VR-protected groups was challenged due to its violation of individual Right to Equality. In September 2022, a compromise that is discussed at the Supreme Court involves maintaining the amendment, but expanding its scope to include the excluded groups. If this compromise is adopted in the country, individuals can belong to multiple VR-protected groups. We show that a major loophole in the system will emerge, if the Supreme Court merely expands the scope of EWS without specifying how its positions are to be processed in relation to earlier caste-based VR-protected positions. Depending on which normative objective the court wants to promote, we formulate and characterize three plausible specifications. If EWS is processed simultaneously with other VR-protected groups, then the outcome is one that selects the most meritorious individuals subject to the Supreme Court's mandates. If EWS is processed before all other VR-protected groups, then the outcome is one that maintains the elevated status of caste-based VR protections. If EWS is processed after all other VR-protected groups, then the outcome is the smallest possible change from the contested amendment that escapes a violation of individual Right to Equality.

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Both Sönmez and Ünver are affiliated with the Department of Economics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA, 02467. Emails: sonmezt@bc.edu, unver@bc.edu. We thank Ashwini Deshpande and Manshu Khanna for their detailed comments.

1. Introduction

Affirmative action is embedded into the 1950 Constitution of India through two types of protective policies, a primary provision called *vertical reservations (VR policy)* and a secondary provision called *horizontal reservations (HR policy)*. On the surface the two policies are similar in that, under both schemes certain percentages of government positions and seats at institutions of higher education are reserved for members of various protected groups. What makes the VR policy a higher-level positive discrimination policy is that, by law VR-protected positions are awarded to members of the protected group who do not merit an unprotected *open-category* position. That is, the reserved positions are not awarded to individuals who are in no need of affirmative action. The HR policy, in contrast, only provides a minimum guarantee to members of the protected group, which means its benefits kick in only of this guaranteed level cannot be reached.

Prior to the 103rd Constitutional Amendment, passed in January 2019, the scope of the VR policy was restricted to members of Scheduled Castes (SCs), Scheduled Tribes (STs), and Other Backward Classes (OBCs), who faced various degrees of social marginalization on the basis of their hereditary caste identity.[1] With a highly controversial amendment to the Constitution, VR protections for up to 10% of government positions and seats at institutions of higher education is now provided for members of a new category called Economically Weaker Sections (EWS). Eligibility for EWS was provided to individuals in financial incapacity, but controversially it was restricted to those who do not qualify for earlier caste-based VR protections. The amendment was immediately challenged at court by several groups, and due to its possible violation of the basic structure of the Constitution it eventually reached to a five-judge Constitution Bench of the Supreme Court in August 2020.

On September 8th, 2022, the Constitution Bench announced the following three main issues for examining whether the amendment violates the basic structure of the Constitution:[2]

(1) Can reservations be granted solely on the basis of economic criteria?
(2) Can states provide reservations in private educational institutions which do not receive government aid?

[1]Scheduled Castes is the official term for Dalits or "untouchables," who endured millenia-long oppression and discrimination due to their lowest status under the caste system. Scheduled Tribes is the official term for the indigenous ethnic groups of India, whose faced oppression due to their isolation and exclusion from mainstream society. Other Backward Classes is the official term that describes lower-level castes who were engaged in various marginal occupation assigned to them by the society to serve castes higher to them in the caste hierarchy.

[2]See the coverage of the case in the *Supreme Court Observer*, last retrieved on 10/01/2022.

(3) Are EWS reservations constitutionally invalid for excluding SCs, STs, and OBCs, from its scope?

In relation to the last issue, advocates for the petitioners repeatedly argued throughout the hearings that the amendment violates the country's *Equality Code* by excluding SCs, STs, OBCs from its scope.[3] Our paper is directly related to this very issue.

On the last day of hearings, as a compromise between the two sides, Dr. Mohan Gopal suggested an alternative way forward that did not involve striking down the amendment. Under this proposal, individuals who are covered by existing VR protections are also included in the scope of the EWS reservations.[4] After hearing both sides' arguments for six and a half days, the Constitution Bench reserved a verdict on these questions on September 27th, 2022.[5] A final decision is expected from the Supreme Court over the next few months.

In this paper, our main objective is exploring how this "compromise" policy–aimed at aligning the contested amendment with India's Equality Code–affects the county's reservation system. Through a series of results, we show that a technical aspect of this compromise policy

(1) fundamentally alters a key feature of the procedures that implement India's reservation policies, and
(2) unless it is addressed, it can result in large-scale unintended consequences.

In our formal analysis in Sections 2-4, we also formulate the tools that are needed to avoid these unintended consequences, and provide three potential paths forward depending on the normative objections of the court.

1.1. Implementation of VR Policy. India's reservation system allocates government positions and seats at public educational institutions to candidates based on their merit scores and their eligibility for VR/HR protections. The ongoing crisis in India is about the VR policy, i.e., the primary affirmative action policy in the country.[6]

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[3]In India, the Equality Code refers to articles 14-18 of the Constitution which formulate the fundamental rights to equality.

[4]See, the document on EWS Reservation Day #8: Responses to Governments Arguments on Last Day of Hearing by the *Supreme Court Observer*, last retrieved on 10/01/2022.

[5]According to Wikipedia, a *reserved* decision is a legal term which judges employ in delaying final judgement for a while.

[6]While any recruitment or admission system implemented in the country has to accommodate the secondary HR policy as well, so far the discussions in the country abstract away from them. In contrast, we consider the HR policy as well in our more general analysis presented in Section 4. However, the HR policy has no bearing on any of the main points we make in this paper. It merely makes the technical analysis of the problem more complex. As such, just as the discussions in the country, we abstract away from the HR policy for our discussions in the Introduction only.

Any time a public institution allocates a number of positions, it has to reserve a legislatively determined percentage for each VR-protected group. The remaining positions are referred to as open category. The term *vertical reservations* was coined in the landmark Supreme Court judgment *Indra Sawhney vs Union Of India (1992)*,[7] which formulated the defining characteristics of this primary protective policy as follows:

- A member of a VR-protected group who deserves an open-category position based on her merit score must be awarded an open-category position, and not deplete the VR-protected positions. VR-protected positions too must be allocated based on merit scores, but they must be saved for those who do not merit an open-category position based on their scores.
- VR-protected positions are *hard reserves* and they are exclusive to members of the protected group.

When no individual is eligible for multiple VR-protected groups, as it has been until now, these two characteristics together imply that the positions should be allocated with the following *Over-and-Above* (O&A) choice rule: First, open-category positions are awarded to individuals with highest merit scores, and next, for each VR-protected category, the protected positions are awarded to remaining members of the category with highest merit scores. Critically, because,

(1) there is no overlap between members of any two VR-protected categories, and
(2) VR-protected positions are exclusively reserved for the members of their respective categories,

it does not matter in what sequence (or other form) the positions reserved for VR-protected categories are allocated under this procedure. That because, provided that both conditions hold, no individual competes for positions at multiple VR-protected categories, thus rendering the competitions at VR-protected categories completely independent from each other. Barring some rare exceptions, both conditions currently hold in field applications in India. Once either condition is relaxed, however, this conclusion no longer holds. In that case allocation of reserved positions at VR-protected categories interfere with each other, thus potentially affecting the distribution of positions. This is why the compromise policy proposed by Dr. Mohan Gopal fundamentally alters a key aspect of the reservation system. Without specification of an additional (and admittedly subtle) aspect of the system (i.e. when EWS positions are to be allocated in relation to positions at other VR-protected categories), a mere expansion of the scope of the EWS reservation to

[7]Widely known as known as the *Mandal Commission Case*, this judgment is considered the main reference for legislation on reservation system. The judgment is available in `https://indiankanoon.org/doc/1363234/`, last retrieved on 03/10/2022.

cover all individuals with financial disability no longer results in a well-defined system under the current legislation.

1.2. Consequences of Overlapping VR Protections. So how important is this new parameter–processing order of the EWS positions in relation to other VR-protected categories–in practice? The short answer to this question is, it is very important! To explain why that is the case, let us consider the following two extreme policies:

(1) *EWS-first VR policy*: EWS positions are processed prior to all other VR-protected positions.
(2) *EWS-last VR policy*: EWS positions are processed after all other VR-protected positions.

Recall that, by legislation open-category positions are allocated prior to all VR-protected positions.

To assess the effect of a potential expansion of the scope of EWS-category eligibility under the *EWS-first VR processing policy*, the following observation is useful: Apart from the exclusion of members of caste-based VR-protected categories, the eligibility conditions for the EWS category is set fairly laxed. According to Deshpande and Ramachandran (2019), 98% of the Indian population earns below the annual income limit Rs 8 lakh to be eligible for the EWS reservation. Hence, it is fairly informative to consider the case where everyone is eligible for the EWS reservation. Well, under this assumption the *EWS-first VR processing policy* is equivalent to completely striking down the EWS reservation! The actual impact will not be exactly identical to this scenario, because in reality not everyone is eligible for the EWS reservation, but the difference will likely be nominal. Thus, under the current income limit from EWS eligibility, the compromise policy has a version that pretty much accounts to eliminating the EWS reservation.[8]

What about the effect of a potential expansion of the scope of EWS-category eligibility under the *EWS-last VR policy*? Remember that, one of the main issues that is to be decided by the Constitution Bench is whether the Right to Equality is violated under the amendment. Assuming that the Bench confirms that the Equality Code is breached in this way, it is formally possible to identify the specific members of SC/ST/OBC who are directly affected by this violation by losing a position they would have received otherwise (Lemma 4). In Theorem 4, we show that under the *EWS-last VR processing policy*,

[8]It may be illustrative to emphasize that, in order that avoid a possible revocation of the EWS reservation, the Central government announced right before the last day of the Supreme Court hearings that, a total of 214,766 additional seats were approved to be created in the central educational institutions, and a total fund of Rs 4,315.15 crores (an equivalent of more than 500 million US dollars) was approved. See *The Print* story for the details.

- this very group of the "compromised" members of SC/ST/OBC replace those from forward castes who each receive a position with lower scores at their expense,
- but otherwise, the rest of the positions are allocated exactly to the same individuals as the current policy.

Thus, the *EWS-last VR processing policy* is literally the smallest possible deviation from the current policy that avoids a violation of the Equality Code. In particular, if no member of SC/ST/OBC loses a position due to their exclusion from the scope of EWS reservation under the current policy, then the outcome of the *EWS-last VR policy* is identical to that of the current policy (Corollary 2). Essentially the *EWS-last VR processing policy* continues to provide its first order benefits to those who are ineligible for other VR-protected categories, but does so in a way that avoids a violation of the Equality Code.

1.3. Normative Implications of Three Focal VR Processing Policies. The contrast between the EWS-first and EWS-last VR-processing policies given above is not meant to be one that endorses one policy or another, but rather an illustration of the nature of the potential loophole that will emerge if the Supreme Court merely expands the scope of EWS eligibility without additional specifications. Indeed, EWS-first or EWS-last VR processing policies are not the only policies that are normatively plausible.

If the normative objective of the court is to maintain neutrality between all VR-protected categories (including EWS) and to enforce a policy that awards the positions to highest merit individuals subject to the earlier base mandates of the Supreme Court, then all VR-protected categories have to be allocated simultaneously through a maximal matching algorithm (Theorem 1). If the normative objective of the court is removing the violation of individual Right to Equality, but otherwise to minimally interfere with the amendment, then EWS positions have to be allocated after all other VR-protected positions (Theorems 2 and 4). If the normative objective of the court is to maintain the elevated status of caste-based VR protections, then EWS positions have to be allocated prior to all other VR-protected positions (Theorems 3 and 5). Given that the normative justifications of these seemingly similar policies are vastly different, we believe it is in the Indian population's best interests to understand their distinction.[9] This is especially the case given the very different distributional implications the EWS-first VR processing policy has compared to the other two policies.

[9]Our exercise here is in the spirit of maintaining *informed neutrality* between reasonable but competing ethical principles (Li, 2017), and minimizing the *normative gap* between intended and implemented normative objectives (Hitzig, 2020).

The rest of this paper is organized as follows: In the next subsection we discuss related literature. In Section 2, we present and analyze the basic version of our model with VR policy only. In Section 3, we present our main application on the contested 103rd Amendment of the Constitution. In Section 4, we present and analyze the general version of our model with VR and HR policies. We conclude in Section 5, and relegate technical preliminaries along with all proofs to the Appendix.

1.4. Related Literature. Starting with Hafalir et al. (2013), there is a growing literature on reserve systems. The role of processing order of different types of positions in this framework was first studied by Kominers and Sönmez (2016) in a theoretical framework, and subsequently by Dur et al. (2018) in the context of school choice. Other papers on reserve systems include Ehlers et al. (2014), Echenique and Yenmez (2015), Dur et al. (2020), Pathak et al. (2020a,b), Abdulkadiroğlu and Grigoryan (2021), Aygün and Bó (2021), Celebi and Flynn (2021, 2022), Celebi (2022), and Sönmez and Yenmez (2022a,b).

Of these papers, four that are especially related to our study are Kominers and Sönmez (2016), Sönmez and Yenmez (2022a), Dur et al. (2018), and Pathak et al. (2020a).

The basic version of our model builds on Kominers and Sönmez (2016) which introduces a general model with slot-specific priorities. The generalized version of our model in Section 4 builds on Sönmez and Yenmez (2022a), which formulates the Indian reservation system with both VR and HR protections, and shows that there is a unique mechanism which satisfies the mandates of the Supreme Court judgments *Indra Sawhney (1992)* and *Saurav Yadav vs The State Of Uttar Pradesh (2020)*.[10] This mechanism, namely the *two-step minimum guarantee* choice rule, is endorsed by the Supreme Court in their judgment *Saurav Yadav (2020)*,[11] and it is further mandated in the state of Gujarat by its high court in *Tamannaben Ashokbhai Desai vs Shital Amrutlal Nishar (2020)*.[12] Our model extends the model of Sönmez and Yenmez (2022a) by allowing for overlaps between VR-protected groups. As we emphasized in Section 1.2, the generalization is highly important in the context of India's contested Constitutional Amendment. Critically, in the absence of additional considerations, the uniqueness result by Sönmez and Yenmez (2022a) no longer holds under our generalization. And as we have thoroughly discussed in Sections 1.2 and 1.3, this observation has major policy implications in relation to the current debates in the country in relation to the contested EWS reservation. In particular, if accepted without

[10]The judgment *Saurav Yadav (2020)* is available in `https://indiankanoon.org/doc/27820739/`, last retrieved on 10/06/2022.

[11]While this mechanism itself is merely endorsed by the Supreme Court, it is de facto mandated due to the uniqueness result by Sönmez and Yenmez (2022a).

[12]The judgment *Tamannaben Ashokbhai Desai (2020)* is available in `https://indiankanoon.org/doc/101656671/`, last retrieved on 10/06/2022.

additional specifications, the compromise policy which extends the scope of EWS reservation will result in a major legislative ambiguity in the country. This potential ambiguity can result both in an erroneous implementation of an unintended policy in the country, and also to its utilization by politically motivated authorities. Interestingly, both types of possibilities transpired in real-life applications of reserve systems.

This type of phenomenon was first presented in Dur et al. (2018) for allocation of seats at Boston Public Schools (BPS) between years 1999-2013. As a compromise between a faction which demanded neighborhood assignment and another which demanded more comprehensive school choice, in 1999 leadership at BPS announced that neighborhood students will receive preferential treatment in half of the seats at each public school. This policy was referred to as *walk-zone priority*. However, processing the walk-zone seats prior to remaining ones effectively negated this policy until 2013. After discovering that their policy was superfluous and misleading, the walk zone policy was abandoned at BPS altogether.

A related phenomenon for allocation of H1-B visas in the US is presented in Pathak et al. (2020a). With the *H-1B Visa Reform Act of 2004*, the US Congress reduced the number of annual H1-B visas from 195,000 to 65,000, but granted an exemption of 20,000 units for holders of advanced degrees. Reflecting a number of purely logistical constraints, the procedure that is used to implement this act was changed a few times over the years, although each time with significant (but likely unrealized) distributional implications. In response to the former President Trump's *Buy American and Hire American Executive Order in 2017*, however, the procedure was reformed by the U.S. Department of Homeland Security with an explicit objective of increasing the number of awards to recipients with advanced degrees. The latest reform, which led to an adoption of a new visa allocation rule for US Fiscal Year 2020, simply involved a reversal of processing sequence of advanced degree visas and general category visas.

Our main formal results are Theorems 1-5. From a technical perspective, the proof Theorem 1 builds on abstract research on matroid theory in Gale (1968), whereas the proof of Theorem 5 closely follows the proof strategy of the main characterization result in Sönmez and Yenmez (2022a). Both the conceptual formulation of Theorem 4 and its proof are novel to our paper. Theorem 2 is a special case of Theorem 4 and Theorem 3 is a special case of Theorem 5.

Finally, more broadly than the literature on reserve systems, there is a large and growing literature on analysis and design of mechanisms which are deployed in settings in which issues of social, racial and distributive justice are particularly important. Our paper contributes to this literature, which includes Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu (2005), Erdil and Ergin (2008), Kesten (2010), Kojima (2012), Pycia (2012), Andersson and Svensson (2014), Kamada and Kojima (2015), Delacrétaz et al. (2016), Chen and Kesten (2017), Fragiadakis and Troyan (2017), Andersson (2019), Erdil and Kumano (2019), Andersson and Ehlers (2020), Ehlers and Morrill (2020), Root and Ahn (2020), and Reny (2022).

2. Model and Preliminary Analysis

There is a finite set $\mathcal{I}$ of individuals who are competing for $q^\Sigma \in \mathbb{N}$ identical positions. Each individual $i \in \mathcal{I}$ is in need of a single position, and has a distinct merit score $\sigma_i \in \mathbb{R}_+$. Let $\sigma = (\sigma_i)_{i \in \mathcal{I}}$ denote the vector of merit scores. In the absence of a positive discrimination policy, individuals with higher merit scores have higher claims for a position. Throughout the paper, we fix the set of all individuals $\mathcal{I}$, the number of positions q^Σ, and the vector of merit scores σ.

There are two types of affirmative action provisions in India: the primary VR policy and the secondary HR policy. We start our analysis by focusing on VR policy only.[13] This more basic version of our model is a refinement of Kominers and Sönmez (2016). An extended model that includes both VR and HR policies, which is a generalization of Sönmez and Yenmez (2022a), is later presented in Section 4.

2.1. Vertical Reservations. Let $\mathcal{R}$ denote the set of **VR-protected categories**. Given an individual $i \in \mathcal{I}$, let $\rho_i \in 2^{\mathcal{R}}$ denote the (possibly empty) set of VR-protected categories she belongs as a member. Let $\rho = \left(\rho_i\right)_{i \in \mathcal{I}} \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ denote the profile of **category memberships**. For each VR-protected category $c \in \mathcal{R}$, let $\mathcal{I}^c(\rho) = \{i \in \mathcal{I} : c \in \rho_i\}$ denote the set of members of category c. We sometimes refer to these individuals as the **beneficiaries** of VR protections at category c. Individuals who do not belong to any VR-protected category are members of a **general category** $g \notin \mathcal{R}$. Let $\mathcal{I}^g(\rho) = \{i \in \mathcal{I} : \rho_i = \emptyset\} = \mathcal{I} \setminus \cup_{c \in \mathcal{R}} \mathcal{I}^c(\rho)$ denote the set of individuals in the general category.

Motivated by the historical structure of VR-protected categories in India, papers in the earlier literature uniformly assume that individual can belong to no more than a single VR-protected category. We drop this assumption, and allow an individual to be member of multiple VR-protected categories. We refer to VR-protected categories as **overlapping**

[13]Focusing on the primary VR policy enables us to relate our analysis to Supreme Court judgments and policy discussions in India, because, the discussions in the country on the 103rd Amendment completely abstract away from the secondary HR policy.

if at least some individuals are members of multiple VR-protected categories, and as **non-overlapping** if each individual is a member of at most one VR-protected category.

For any VR-protected category $c \in \mathcal{R}$, $q^c \in \mathbb{N}$ positions are exclusively set aside for the members of category c.[14] These provisions are referred to as **VR-protected positions at category c**. For any VR-protected category $c \in \mathcal{R}$, let $\mathcal{E}^c(\rho) = \mathcal{I}^c(\rho)$ denote the set of individuals who are **eligible for VR-protected positions at category c**. The total number of all VR-protected positions is no more than the number of all positions. That is,

$$\sum_{c \in \mathcal{R}} q^c \leq q^{\Sigma}.$$

The remaining

$$q^o = q^{\Sigma} - \sum_{c \in \mathcal{R}} q^c$$

positions are referred to as **open-category** (or **category-o**) positions. Let $\mathcal{E}^o(\rho) = \mathcal{E}^o = \mathcal{I}$ denote the set of individuals who are **eligible for open-category positions**.

Let $\mathcal{V} = \mathcal{R} \cup \{o\}$ denote the set of **vertical categories for positions**.

2.2. Solution Concepts and Primary Axioms. We next present the solution concepts used in our paper, and the primary axioms imposed on them. Throughout this section, fix a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$.

Definition 1. Given a category $v \in \mathcal{V}$, a **single-category choice rule** is a function $C^v\Big(\rho; .\Big) : 2^{\mathcal{I}} \to 2^{\mathcal{I}}$, such that, for any set of individuals $I \subseteq \mathcal{I}$,

$$C^v\Big(\rho; I\Big) \subseteq I \bigcap \mathcal{E}^v(\rho) \quad \text{and} \quad \Big|C^v\Big(\rho; I\Big)\Big| \leq q^v.$$

That is, for any set of individuals, a single-category choice rule selects a subset from those eligible, and up to capacity.

Definition 2. A **choice rule** is a multidimensional function $C(\rho; .) = \big(C^{\nu}(\rho; .)\big)_{\nu \in \mathcal{V}} : 2^{\mathcal{I}} \to (2^{\mathcal{I}})^{|\mathcal{V}|}$ such that, for any set of individuals $I \subseteq \mathcal{I}$,

(1) for any category $v \in \mathcal{V}$,

$$C^v(\rho; I) \subseteq I \bigcap \mathcal{E}^v(\rho) \quad \text{and} \quad |C^v(\rho; I)| \leq q^v,$$

(2) for any two distinct categories $v, v' \in \mathcal{V}$,

$$C^v(\rho; I) \bigcap C^{v'}(\rho; I) = \emptyset.$$

That is, a choice rule is a list of interconnected single-category choice rules for each category of positions, where no individual is selected by more than a single category.

[14]This type of protective policy is sometimes referred to as *hard reserves*.

Definition 3. For any choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu \in \mathcal{V}}$, the resulting **aggregate choice rule** $\widehat{C}(\rho;.) : 2^{\mathcal{I}} \to 2^{\mathcal{I}}$ is given as, for any $I \subseteq \mathcal{I}$,

$$\widehat{C}(\rho; I) = \bigcup_{v \in \mathcal{V}} C^{v}(\rho; I).$$

For any reservation vector, profile of category memberships, and set of individuals, the aggregate choice rule yields the set of chosen individuals across all categories.

As it is discussed in depth in Sönmez and Yenmez (2022a), the following three axioms are mandated throughout India with the Supreme Court judgment *Indra Sawhney (1992)*.[15] In our main analysis, we focus on choice rules that satisfy all three axioms.

Definition 4. A choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu \in \mathcal{V}}$ satisfies **non-wastefulness** if, for any $I \subseteq \mathcal{I}$, $v \in \mathcal{V}$, and $j \in I$,

$$j \notin \widehat{C}(\rho; I) \text{ and } |C^{v}(\rho; I)| < q^{v} \quad \implies \quad j \notin \mathcal{E}^{v}(\rho).$$

The first axiom requires no position to remain idle for as long as there is an eligible individual.

Definition 5. A choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu \in \mathcal{V}}$ satisfies **no justified envy** if, for any $I \subseteq \mathcal{I},\ v \in \mathcal{V},\ i \in C^{v}(\rho; I)$, and $j \in \big(I \cap \mathcal{E}^{v}(\rho)\big) \setminus \widehat{C}(\rho; I)$,

$$\sigma_i > \sigma_j.$$

The second axiom requires that no individual receives a position at any category $v \in \mathcal{V}$ at the expense of another individual who is both eligible for the position and at the same time has higher merit score.

Definition 6. A choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu \in \mathcal{V}}$ satisfies **compliance with VR protections** if, for any $I \subseteq \mathcal{I}$, $c \in \mathcal{R}$, and $i \in C^{c}(\rho; I)$,

(1) $|C^{o}(\rho; I)| = q^{o}$, and
(2) for every $j \in C^{o}(\rho; I)$,

$$\sigma_j > \sigma_i.$$

The third axiom requires that, an individual who is "deserving" of an open-category position due to her merit score, should be awarded an open-category position and not one that is VR-protected.

[15] See Section 4.2 for generalizations of these axioms (along with a fourth axiom) for the extended model with both protective policies, each one formulating the more refined mandates of the Supreme Court judgment *Saurav Yadav (2020)*.

2.3. Sequential Choice Rules. In this section we present *sequential* choice rules, a class of rules that plays a prominent role in most real-life applications of reserve systems. The name of the class captures the idea that, under each member of the class, all positions in any category (including the open category) are processed in blocks following a given sequence of categories.

Let Δ denote the set of all linear orders on $\mathcal{V}$. Each element of Δ represents a linear processing sequence of vertical categories, and referred to as an **order of precedence**. Let

- $\Delta^o \subsetneq \Delta$ denote all sequences where the open-category is processed first.

For any VR-protected category $c \in \mathcal{R}$, let

- $\Delta^o_c \subsetneq \Delta$ denote all sequences where the open-category is processed first and category c is processed last, and
- $\Delta^{o,c} \subsetneq \Delta$ denote all sequences where the open-category is processed first and category c is processed second.

Next, given a category $v \in \mathcal{V}$, define the following single-category choice rule: For any set of individuals $I \subseteq \mathcal{I}$, the **category-v serial dictatorship** $C^v_{sd}(\rho;.)$ selects the set of highest merit-score individuals in I who are eligible for category v, up to capacity.

Fix an order of precedence $\rhd \in \Delta$. Given a set of individuals $I \subseteq \mathcal{I}$, the outcome of the sequential choice rule $C_S(\rhd,\rho;.)$ is obtained with the following procedure.

Sequential Choice Rule $C_S(\rhd,\rho;.) = \big(C^\nu_S(\rhd,\rho;.)\big)_{\nu\in\mathcal{V}}$

Step 0 (Initiation): Let $I_0 = \emptyset$.
Step k (**$k \in \{1,\ldots,|\mathcal{V}|$**): Let v_k be the category which has the k^{th} highest order of precedence under $\rhd$.

$$C^{v_k}_S(\rhd,\rho;I) = C^{v_k}_{sd}\Big(\rho;\big(I \setminus I_{k-1}\big) \cap \mathcal{E}^{v_k}\Big)$$

Let $I_k = I_{k-1} \cup C^{v_k}_S(\rhd,\rho;I)$.

For each vertical category $v \in \mathcal{V}$, the category-v serial dictatorship $C_{sd}(\rho;.)$ is applied sequentially under this procedure, following their order of precedence under $\rhd$.

2.3.1. *Relation to Over-and-Above Choice Rule and Preliminary Results on Sequential Choice Rules.* Over-and-Above (O&A) choice rule $C_{OA}(\rho;.) = \big(C^\nu_{OA}(\rho;.)\big)_{\nu\in\mathcal{V}}$ is a special case of a sequential choice rule in that,

(1) it is defined only for environments with non-overlapping VR protections, and
(2) it processes the open category before any other category.

Thanks to these two restrictions, the relative processing sequence of the VR-protected categories becomes completely immaterial under the O&A.[16]

The following straightforward characterization, first formally stated in Sönmez and Yenmez (2022a), is the starting point of our analysis.

Proposition 0. *Fix a profile of category memberships* $\rho = (\rho_i)_{i\in\mathcal{I}} \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *such that VR-protected categories are non-overlapping. Then a choice rule* $C(\rho;.)$ *satisfies non-wastefulness, no justified envy, and compliance with VR protections if and only if* $C(\rho;.) = C_{OA}(\rho;.)$.

Therefore, O&A choice rule uniquely satisfies the mandates of the Supreme Court judgment *Indra Sawhney (1992)* for as long as the VR protections are non-overlapping (as it has historically been).

As we thoroughly discussed in the Introduction, however, the current non-overlapping structure of the VR-protected groups may potentially lost in the country in the near future if the Supreme Court extends eligibility for the EWS reservation.[17] This observation renders other sequential choice rules to be of potential interest.

The following lemmata show that each sequential choice rule satisfies the first two axioms mandated by *Indra Sawhney (1992)*, and for as long as it processes the open-category before other categories, it also satisfies the third axiom.

Lemma 1. *For any* $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *and* $\rhd \in \Delta$, *the sequential choice rule* $C_S(\rhd, \rho;.)$ *satisfies non-wastefulness and no justified envy.*

Lemma 2. *For any* $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *and* $\rhd \in \Delta^o$, *the sequential choice rule* $C_S(\rhd, \rho;.)$ *satisfies compliance with VR protections.*

The next lemma further shows that, even if non-regular VR policies are adopted at some (or all) of potentially overlapping VR-protected categories, any choice rule that satisfies the Supreme Court's mandates still has to allocate the open-category positions to the same individuals who would have received them under O&A. Thus, any deviation from the outcome of the O&A choice rule is due to the assignment of VR-protected positions.

Lemma 3. *Fix a profile* $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *of category memberships, and another a profile* $\rho' \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *of category memberships that is non-overlapping. Let* $C(\rho;.) = \left(C^{\nu}(\rho;.)\right)_{\nu\in\mathcal{V}}$ *be any choice rule that satisfies non-wastefulness, no justified envy, and compliance with VR protections. Then, for any set of individuals* $I \subseteq \mathcal{I}$,

$$C^o(\rho; I) = C^o_{OA}(\rho'; I).$$

[16]Since the outcome of the O&A choice rule is independent of the choice of an order of precedence in Δ^o, the parameter $\rhd$ is suppressed in $C_{OA}(\rho;.)$.

[17]See Section 3 for details.

2.4. Meritorious Over-and-Above Choice Rule. As presented in Section 2.3, sequential choice rules abide by the mandates of *Indra Sawhney (1992)*. Despite the prominence of these rules in real-life applications, however, they are not the only viable extensions of the O&A choice rule when VR protections are allowed to be overlapping. On the contrary, absent of additional criteria, there is another choice rule which is "superior" to any other for this more general version of the problem. In this section, we introduce and analyze this better alternative that refer to as the *meritorious Over-and-Above* (mO&A) choice rule.

Throughout this section, we fix a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$.

Definition 7. Let $\beta : 2^{\mathcal{I}} \to \mathbb{N}$ denote the **VR-maximality function** that gives the maximum number of VR-protected positions that can be awarded to eligible individuals in I, for any set of individuals $I \subseteq \mathcal{I}$.

For any set of individuals $I \subseteq \mathcal{I}$, this number is simply given as

$$\beta(I) = \sum_{c \in \mathcal{R}} \min \left\{ |I \cap \mathcal{E}^c(\rho)|, q^c \right\},$$

when the VR protections are non-overlapping. For the general case with overlapping VR protections, we define it formally in Definition A.4 in Appendix A.

The following auxiliary definition simplifies the formulation of the mO&A choice rule.

Definition 8. For any set of individuals $I \subseteq \mathcal{I}$ and an individual $i \in \mathcal{I} \setminus I$ (who is not a member of set I), **individual i increases the VR-utilization of I** (upon joining individuals in set I) if,

$$\beta(I \cup \{i\}) = \beta(I) + 1.$$

Given a set of individuals $I \subseteq \mathcal{I}$, the outcome of the mO&A choice rule $C_{\text{OA}}(\rho;.)$ is obtained in two steps with the following procedure.

Meritorious Over-and-Above Choice Rule $C_{\text{OA}}(\rho;.) = \left(C^{\nu}_{\text{OA}}(\rho;.)\right)_{\nu \in \mathcal{V}}$

Step 1. Open category is processed. Choose the highest merit score individuals in I one at a time until $\min\{|I|, q^o\}$ individuals are chosen. Let $C^o_{\text{OA}}(\rho; I)$ be the set of chosen individuals.

Step 2. All VR-protected categories are processed in parallel.
Let $J = I \setminus C^o_{\text{OA}}(\rho; I)$ be the set of remaining individuals in I.

Step 2.0 (Initiation): Let $J_0 = \emptyset$.

Step 2.k ($k \in \{1, \ldots, \sum_{c \in \mathcal{R}} q^c\}$): Assuming such an individual exists, choose the highest merit score individual in $J \setminus J_{k-1}$ who increases the VR-utilization of J_{k-1}. Denote this individual by j_k and let $J_k = J_{k-1} \cup \{j_k\}$. If no such individual exists, then end the process.

For any $c \in \mathcal{R}$, let $C^{c}_{\text{(OA)}}(\rho; I)$ be the set of individuals who each received a category-c position in this step.

Remark 1. While the set of individuals $\widehat{C}_{\text{(OA)}}(\rho; .)$ who are chosen under the above-given procedure is uniquely defined, there may be multiple ways to assign some individuals to their VR-protected categories. This potential multiplicity is benign for our analysis.

The mO&A choice rule abides by the mandates of *Indra Sawhney (1992)*.

Proposition 1. *The meritorious Over-and-Above choice rule $C_{\text{(OA)}}(\rho; .)$ satisfies non-wastefulness, no justified envy, and compliance with VR protections.*

We need the following terminology to present our first main result.

Definition 9 (Gale (1968))**.** Let members of two sets of individuals $I = \{i_1, \ldots, i_{|I|}\}, J = \{j_1, \ldots, j_{|J|}\} \subseteq \mathcal{I}$ be each enumerated such that the higher the merit score of an individual is the lower index number she has. Then, the set of individuals I **Gale dominates** the set of individuals J if,

(1) $|I| \geq |J|$, and

(2) for each $\ell \in \{1, \ldots, |J|\}$,
$$\sigma_{i_\ell} \geq \sigma_{j_\ell}.$$

When a set of individuals I Gale dominates another set of individuals J,

(1) there are at least as many individuals who are admitted under I as under J, and
(2) the highest merit-score individual in I is at least as meritorious as the highest merit-score individual in J, the second highest merit-score individual in I is at least as meritorious as the highest merit-score individual in J, and so on.

Thus, I as a group is more meritorious than J in a very clear and strong sense. Our first main result establishes that, while the uniqueness of a choice rule that satisfies the Supreme Court's mandates on *Indra Sawhney (1992)* is lost once VR protections are overlapping, there is still a choice rule that is better than any other for a system that promotes meritocracy.

Our first main result justifies the naming of the meritorious Over-and-Above choice rule.

Theorem 1. *Let $C(\rho; .)$ be any choice rule that satisfies non-wastefulness, no justified envy, and compliance with VR protections. Then for any set of individuals $I \subseteq \mathcal{I}$, the set of individuals $\widehat{C}_{\text{(OA)}}(\rho; I)$ admitted by the meritorious Over-and-Above choice rule Gale dominates the set of individuals $\widehat{C}(\rho; I)$ admitted under choice rule $C(\rho; .)$.*

Given the mandates of *Indra Sawhney (1992)* and Theorem 1, it may be tempting to declare the meritorious Over-and-Above choice rule as the unambiguous "winner" for the more general version of the problem with overlapping VR protections. Nevertheless, we caution against a hasty dismissal of other alternatives for there may be application-specific considerations which may deem other choice rules as strong contenders. Focusing on the specifics of the pending Supreme Court case on the EWS reservation, we next present two other generalizations of the O&A choice rule that also deserve serious consideration.

3. Analysis for the Supreme Court Case on EWS Reservation

Until recently, VR protections in India had been exclusive to certain groups that suffered marginalization and discrimination due to their caste identities. This norm has changed with the 103rd Amendment of the Constitution of India, which came into effect in January 2019. Under the amendment, the *Economically Weaker Sections* of the society are awarded with VR protections for up to 10 percent of the positions in government jobs and seats in higher education on the basis of their financial incapacity. Importantly, the non-overlapping structure of the VR protections is maintained with the amendment, and groups who are eligible for the existing VR protections, i.e. SCs, STs, and OBSs, are excluded from the new provisions. Accordingly, the new VR-protected group is officially identified as "Economically Weaker Section in the General category."

In January 2019, Youth For Equality, an NGO that opposes caste-based policies, challenged the amendment at the Supreme Court. Among their main objections is the exclusion of members of SCs, STs, and OBCs from the new EWS category, as it violates the Constitutional *Right to Equality*. Consequently, the present non-overlapping structure of the VR-protected groups is also under dispute with the challenge of the amendment, which has advanced to a five-judge Constitution Bench of the Supreme Court in August 2020. On September 8th, 2022, the Bench announced the following issue as one of the three that will be decided: "If EWS reservations are invalid for excluding SCs, STs, OBSs from its scope?" Hence, the Bench will decide whether the Constitutional Right for Equality is violated for members of SCs, STs, and OBSs. Following seven days of hearings, the arguments from both sides concluded on September 27th, and the Constitution Bench reserved its judgment.[18]

Our formal analysis in this section is based on the premise that, exclusion of SCs, STs, and OBSs from the scope of EWS reservation results in a violation for the Right to Equality.

[18]The current status of the case can be found in the following link `https://www.scobserver.in/cases/youth-for-equality-union-of-india-ews-reservation-case-background/`.

Under this premise, this exclusion has to be removed regardless of which generalization of the O&A choice rule is adopted in the field. In this context, Theorem 1 can be interpreted as a justification for processing all VR categories simultaneously. In Sections 3.1 and 3.2, in contrast, we present potential justifications for processing the EWS category after caste-based VR-protected categories and before caste-based VR-protected categories respectively.

Since this section pertains to the crisis on EWS reservation, we focus on a version of the model where there is a single VR-protected category $e \in \mathcal{R}$ that corresponds to EWS and shares common members with other VR-protected categories. Let $\mathcal{R}^0 = \mathcal{R} \setminus \{e\}$ denote the set of caste-based VR-protected categories. Thus, categories in $\mathcal{R}^0$ are non-overlapping among themselves, but they each overlap with category e.

3.1. The Case for the EWS-last VR Processing Policy. In this section we consider a policy that removes the exclusion of caste-based VR categories from the scope of EWS, and processes EWS category after all other categories. The first part of this reform directly addresses the (potential) violation of the Right to Equity, and it is the same as Dr. Gopal's proposal in the last day of the hearings of the ongoing case on EWS reservation. Therefore, through the second part of this reform, a refinement of Dr. Gopal's proposal is presented. This policy merits serious consideration for it avoids a violation of Right to Equality though the smallest possible interference with the existing Constitutional Amendment. In order to present why that is the case, we need the following additional analysis.

3.1.1. *Who Are Directly Affected from the Violation of Right to Equality?* Let $\mathring{\rho} = (\mathring{\rho}_i)_{i\in\mathcal{I}} \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ denote the original (i.e. existing) profile of category memberships in the absence of overlapping VR protections. Therefore, $|\mathring{\rho}_i| \leq 1$ for each $i \in \mathcal{I}$. Fix a set of individuals $\mathcal{J} \subseteq \bigcup_{c\in\mathcal{R}^0}\mathcal{E}^c(\mathring{\rho}) \subseteq \mathcal{I}$ whose Right to Equality is violated due to exclusion from the scope of EWS reservation.

Consider the following question: Who among individuals in $\mathcal{J}$ can argue that she is materially affected by the violation of her Right to Equality, because she lost a position due to her exclusion from the scope of category e under $\mathring{\rho}$? An individual $i \in \mathcal{J}$ can make this argument, if she remains unmatched under the membership profile $\mathring{\rho}$, although she would have received a position under an alternative scenario where she is granted with a membership of the new category e instead of a membership of her existing category in $\mathcal{R}^0$. This observation motivates the following series of definitions.

Given an individual $j \in \mathcal{J}$, let $\tilde{\rho}_j = \{e\}$.

Definition 10. Given a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$, a choice rule $C(\rho; .)$, and a set of individuals $I \subseteq \mathcal{I}$, an individual $j \in \mathcal{J} \bigcap I$ **suffers from a violation of the**

Equality Code under $C(\rho;.)$ for I, if

$$j \notin \widehat{C}\Big(\rho; I\Big) \quad \text{and} \quad j \in \widehat{C}\Big(\big(\rho_{-j}, \tilde{\rho}_j\big); I\Big).$$

Given a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$, a choice rule $C(\rho;.)$, and a set of individuals $I \subseteq \mathcal{I}$, a set of individuals $J \subseteq \mathcal{J} \cap I$ **suffer from a violation of the Equality Code** under $C(\rho;.)$ for I, if, for each $j \in J$,

$$j \notin \widehat{C}\Big(\rho; I\Big) \quad \text{and} \quad j \in \widehat{C}\Big(\big(\rho_{-J}, \tilde{\rho}_J\big); I\Big).$$

Definition 11. Given a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ and a choice rule $C(\rho;.)$, a set of individuals $I \subseteq \mathcal{I}$ are **materially unaffected by the violation of the Equality Code** under the choice rule $C(\rho;.)$, if there exists no set of individuals $J \subseteq \big(\mathcal{J} \cap I\big)$ who suffer from a violation of the Equality Code under $C(\rho,.)$ for I.

By definition, for any set of individuals $J \subseteq \mathcal{J} \cap I$ who suffer from a violation of the Equality Code under the choice rule $C(\rho;.)$ for I, we have $J \cap \widehat{C}(\rho; I) = \emptyset$.

Definition 12. Given a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$, a choice rule $C(\rho;.)$ **abides by the Equality Code**, if, for any set of individuals $I \subseteq \mathcal{I}$, there exists no set of individuals $J \subseteq \big(\mathcal{J} \cap I\big)$ who suffer from a violation of the Equality Code under $C(\rho,.)$ for I.

By definition, for any set of individuals $J \subseteq \mathcal{J} \cap I$ who suffer from a violation of the Equality Code under the choice rule $C(\rho;.)$ for I, we have $J \cap \widehat{C}(\rho; I) = \emptyset$.

Observe that, a set of individuals may suffer from a violation of the Equality Code based on Definition 10, and yet some of its members may still not be deserving of a position, because there may be other individuals who also suffer from a violation of the Equality Code despite being even more meritorious. This observation motivates our next definition.

Definition 13. Given a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$, a choice rule $C(\rho;.)$, and a set of individuals $I \subseteq \mathcal{I}$, the set of individuals $J \subseteq \Big(\mathcal{J} \cap I\Big) \setminus \widehat{C}(\rho; I)$ is a **maximal set of individuals who suffer from a violation of the Equality Code** under $C(\rho;.)$ for I, if,

(1) the set of individuals J suffer from a violation of the Equality Code under $C(\rho;.)$ for I, and
(2) for any set of individuals $J' \subseteq \Big(\mathcal{J} \cap I\Big) \setminus \widehat{C}(\rho; I)$ with $J \subsetneq J'$,
    (a) J' does not suffer from a violation of the Equality Code under $C(\rho;.)$ for I, and
    (b) $J \subseteq \widehat{C}\Big(\big(\rho_{-J'}, \tilde{\rho}_{J'}\big); I\Big)$.

Lemma 4. *Fix the profile of category memberships as $\mathring{\rho}$. For any set of individuals $I \subseteq \mathcal{I}$, the maximal set of individuals who suffer from a violation of the Equality Code under the choice rule $C_{OA}(\mathring{\rho};.)$ for I is uniquely defined.*

Lemma 4 implies that, the maximal set of individuals who can argue that they each lost a position because the Constitutional Amendment excluded them from the scope of the EWS category is uniquely defined.

3.1.2. *EWS-last Over-and-Above Choice Rule.* We are ready to formulate an sequential choice rule that not only abides by the Equality Code, but it is also a minimal deviation from the existing system in a well-defined sense.

Let $\rho^* = (\rho^*_i)_{i\in\mathcal{I}}$ be such that,

(1) $\rho^*_i = \mathring{\rho}_i \cup \{e\}$ for any $i \in \mathcal{J}$, and
(2) $\rho^*_i = \mathring{\rho}_i$ for any $i \in \mathcal{I} \setminus \mathcal{J}$.

Compared to the profile of category memberships $\mathring{\rho}$, the adjusted profile of category memberships ρ^* grants each member of set $\mathcal{J}$ an extra membership of category e. Since each of these individuals are already member of a caste-based VR-protected category, the structure of category memberships under this modification involves overlapping VR protections. Moreover, it also means that, the O&A choice rule is no longer well defined. Therefore, instead, we propose a sequential choice rule where the order of precedence is such that the open-category is processed first and the "scope-extended" VR-protected category e the last.

Note that, the first part of the resulting policy involves an increase in the scope of EWS category is parallel to the proposal of Dr. Mohan Gopal that was brought to the Constitution Bench during the last day of hearings for the pending case. Second part of the policy refines Dr. Gopal's proposal by processing the scope-extended EWS category after all other categories. We refer to this second (and more subtle) aspect of the resulting policy as **EWS-last VR processing policy**.

We next present, why this policy represents the smallest possible change from the contested amendment for an authority whose objective is simply avoiding a violation of the Equality Code.

Let $\trianglerighteq \in \Delta^o_e$, i.e., the order of precedence $\trianglerighteq$ orders the open category first and category e last. As an alternative choice rule that abides by the Equality Code through a "minimal interference" with the existing system, we formulate a sequential choice rule that is induced by the amended category membership profile ρ^* along with the order of precedence $\trianglerighteq$.[19] In the context of the crisis on EWS reservation, we refer to this rule as the **EWS-last Over-and-Above** (EWS-last O&A) choice rule.

The EWS-last O&A choice rule not only satisfies the mandates of the Supreme Court in *Indra Sawhney (1992)*, but it also abides by the Equality Code.

Proposition 2. *Fix the profile of category memberships as ρ^* and the order of precedence as $\trianglerighteq \in \Delta_e^o$. Then, the sequential choice rule $C_S(\trianglerighteq, \rho^*; .)$ abides by the Equality Code and it satisfies non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections.*

We are ready to present the main result of this section.

Theorem 2. *Consider any set of individuals $I \subseteq \mathcal{I}$. Then, the set of individuals*

$$C_S(\trianglerighteq, \rho^*; I) \setminus C_{OA}(\mathring{\rho}; I)$$

is equal to the maximal set of individuals who suffer from a violation of the Equality Code under the choice rule $C_{OA}(\mathring{\rho}; .)$ for I.

Corollary 1. *Consider any set of individuals $I \subseteq \mathcal{I}$. Then, we have*

$$C_S(\trianglerighteq, \rho^*; I) = C_{OA}(\mathring{\rho}; I)$$

if and only if the set of individuals I are materially unaffected by the violation of the Equality Code under the choice rule $C_{OA}(\mathring{\rho}; .)$.

Theorem 2 states that the outcome of the EWS-last O&A choice rule differs from the outcome of the existing O&A choice rule only if the latter involves a violation of the Equality Code, and when its outcome differs from the current rule, it does so by merely replacing the beneficiaries of this oversight with those who suffer from the violation of the Equality Code. In that sense this particular reform can be considered one that "minimally interferes" with the current system. Whereas the appeal of the meritorious Over-and-Above choice rule–presented in Section 2.4–is rooted in fundamental principles, the "proximity" of the EWS-last O&A choice rule to the existing system is its main appeal.

3.1.3. *Conditional Membership for the New Members Under the EWS-last VR Processing Policy.* It is worthwhile to highlight the important role the specific order of precedence $\trianglerighteq \in \Delta_e^o$ plays under the sequential choice rule $C_S(\trianglerighteq, \rho^*; .)$. Under this order of precedence,

[19] Here, the outcome of the resulting choice rule is independent of which order of precedence is picked from Δ_e^o, and therefore the formulation corresponds to a unique choice rule.

(1) positions in the open-category are allocated prior to positions in any other category (as it is mandated under under *Indra Sawhney (1992)*),
(2) but more critically, the positions in category e are processed after the positions in all other categories.

This selection has a key implication on the scope of benefits the extra category-e membership given to members of the set $\mathcal{J} \subseteq \cup_{c \in \mathcal{R}^0} \mathcal{E}^c(\mathring{\rho})$ under expanded category memberships in ρ^*. More specifically, the potential benefits of this new membership becomes largely diminished for individuals in $\mathcal{J}$ under the order of precedence $\underline{\triangleright}$, and the benefits kick in only if there would be a violation of the Equality Code in the absence of their category-e membership.[20] That is because, since positions at all other categories are already allocated prior to allocation of positions in category e under the order of precedence $\underline{\triangleright}$, relatively the lower merit-score members of other VR-protected categories remain in competition for allocation of category-e positions. And if there wouldn't be any violation of the Equality Code in the absence of the extra memberships provided under ρ^*, then all these positions are awarded to existing members of category e under $\tilde{\rho}$. Therefore, the extra membership provided to members of set $\mathcal{J}$ under the sequential choice rule $C_S(\underline{\triangleright}, \rho^*; .)$ can be interpreted as a "conditional membership" which only kicks in when it is absolutely necessary to avoid a violation of the Equity Code. This important observation is also the main driving force behind Theorem 2.

3.2. The Case for the EWS-first VR Processing Policy. So far we considered two choice rules under the scope-extended EWS category: The meritorious Over-and-Above choice rule under which all VR-protected categories are processed simultaneously, and the EWS-last Over-and-Above choice rule under which category EWS is processed after all other VR-protected categories. As a third policy that also deserves serious consideration under the scope-extended EWS category, we next consider one final policy that processes EWS category immediately after the open-category. We refer to this policy as the **EWS-first VR processing policy**.

Let $\overline{\triangleright} \in \Delta^{o,e}$, i.e., the order of precedence $\overline{\triangleright}$ orders the open category first and category e next. As a third and final choice rule that deserves serious consideration to resolve the ongoing crisis on EWS reservation, we formulate a sequential choice rule that is induced by the expanded category membership profile ρ^* given in Section 3.1, along with the

[20]In contrast, the potential benefits would always fully kick in under an alternative order of precedence $\overline{\triangleright} \in \Delta^{o,e}$, where category e positions are allocated immediately after the open-category positions. See Section 3.2 for a justification for this alternative policy.

order of precedence $\overline{\triangleright}$.[21] In the context of the crisis on EWS reservation, we refer to this rule as the **EWS-first Over-and-Above** (EWS-first O&A) choice rule.

3.2.1. *Should the Elevated Status of Caste-Based VR Protections be Maintained?* As we have emphasized before, the VR policy is intended as the strongest form of positive discrimination policy in India. The reason such a powerful policy is adopted is historical. The VR policy was originally intended for members of Scheduled Castes which is the official term for Dalits or "untouchables," whose members have suffered millennia-long systematic injustice due to their lowest status under the caste system, and Scheduled Tribes, which is the official term for the indigenous ethnic groups of India, whose members were both physically and socially isolated from the rest of the society. With *Indra Sawhney (1992)*, VR policy was also extended to members of Other Backward Classes who were historically engaged in various marginal occupation assigned to them by the society to serve castes higher to them in the caste hierarchy. The strength of the regular VR protection policy reflects the strong desire in Indian society to acknowledge and reverse the excessively disadvantaged status of these communities.

But other than the fraction of positions reserved under the VR policy, which other aspects of this policy makes it especially powerful? The answer to this question lays in the following two technical aspects of this policy. First of all, positions that are set aside for a VR-protected category are exclusive to its beneficiaries, even when it means that some of these positions remain unassigned.[22] But more importantly, the landmark Supreme Court judgment *Indra Sawhney (1992)* explicitly mandated that these positions are not to be used for members of the VR-protected category who could already receive a unit from open category without invoking the benefits of VR policy (i.e., the axiom *compliance with VR policy*). Thus, the VR-protected positions are explicitly directed to those who could not receive an open position. It is this second aspect of the VR policy that makes it especially powerful.

A key question that needs to be answered by members of the judiciary and the policymakers in India is the following: What happens if a financially disadvantaged member of a caste-based VR category (say SC) does not merit a position from the open category, but she merits a position both from EWS and also SC? Should the principle that requires to allocate the open-category positions before the SC positions also be applied between EWS positions and SC positions? Equivalently, should an SC position be awarded to a

[21]As in the case of the EWS-last Over-and-Above choice rule, the outcome of the resulting choice rule is independent of which order of precedence is picked from $\Delta^{o,e}$, and therefore the formulation corresponds to a unique choice rule.

[22]These unassigned units are typically carried over to the next allocation period to be added to the VR-protected positions for the same category.

financially-disadvantaged member of SC who already merits an EWS position or not? Since the Prime Minister Narendra Modi have repeatedly asserted that "the EWS reservation in the general category will not undermine the interests of dalits, tribals and the Other Backward Classes," it may be natural to answer this important question in the negative.[23] If the members of the judiciary and the policymakers in India agree with this conclusion, then the following additional axiom should also be imposed on a choice rule.

Definition 14. A choice rule $C(\rho;.) = \left(C^{\nu}(\rho;.)\right)_{\nu\in\mathcal{V}}$ **maintains the elevated status of caste-based VR protections** if, for any $I \subseteq \mathcal{I}$, $c \in \mathcal{R}^0$, and $i \in C^c(\rho; I) \bigcap \mathcal{E}^e(\rho)$,

(1) $|C^e(\rho; I)| = q^e$, and
(2) for every $j \in C^e(\rho; I)$,
$$\sigma_j > \sigma_i.$$

Our next result states that, EWS-first Over-and-Above choice rule is the only rule that satisfies this additional axiom along with the mandates of *Indra Sawhney (1992)*.

Theorem 3. *Fix a profile of category memberships* $\rho = (\rho_i)_{i\in\mathcal{I}} \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *such that, for any* $i \in \mathcal{I}$,
$$|\rho(i)| \leq 2 \quad \textit{and} \quad |\rho(i)| = 2 \implies e \in \rho(i).$$
Fix an order of precedence as $\overline{\rhd} \in \Delta^{o,e}$. *Then, a choice rule* $C(\rho;.)$ *maintains the elevated status of caste-based VR protections and it satisfies non-wastefulness, no justified envy, and compliance with VR protections if and only if* $C(\rho;.) = C_S(\overline{\rhd}, \rho;.)$.

It is important to emphasize one important aspect of the income limit for EWS eligibility, and its impact on the EWS-first O&A choice rule. While EWS category intended for individuals who are financially disabled, according to Deshpande and Ramachandran (2019), 98% of the Indian population earns below the annual income limit Rs 8 lakh to be eligible for the EWS reservation. Therefore, under the current disputed scope of the EWS category, the Constitutional Amendment essentially provided a vertical category for members of forward castes. If, however, the exclusion of SEBSs from the scope of the EWS is revoked, then the current income limit for EWS eligibility has a completely orthogonal implication under the EWS-first O&A choice rule. The outcome of the EWS-first O&A choice rule is virtually same as a policy that completely removes the EWS category! In particular, if everyone were to be eligible for EWS (rather than the current 98%), then the outcome of the EWS-first O&A choice rule is exactly the same as transferring all EWS positions to open category!

[23] See, for example, *The Print* story "EWS quota will not affect existing reservation, says PM Modi," last retrieved on 10/30/2022.

3.3. Legislative Loophole under a Reform Which Only Extends the Scope of EWS Category. So what happens if justices resolve the ongoing crisis by merely increasing the scope of EWS reservation, but without interfering with the processing sequence of the EWS positions in relation to other VR-protected positions? We view this possibility as a likely but undesirable scenario, for it will introduce a major loophole in the system.

So which extension of the O&A choice rule has to be adopted in the country? Our purpose in this paper is not advocating for a specific choice rule, but rather formulating some of the most natural alternatives and charactering their policy implications. If the objective is to maintain neutrality between all VR-protected categories, then meritorious O&A choice rule should be adopted with the recognition that this sequencing will tilt the EWS category in favor of forward castes. If the objective is to remove the violation of Right to Equality through the smallest possible modification of the amendment, then the EWS-last O&A choice rule should be adopted with the recognition that this choice rule too will tilt the EWS category in favor of forward castes. If the objective is to maintain the elevated status of caste-based VR protections, then EWS-first O&A choice rule should be adopted with the recognition that, under the current income limit for EWS, this choice rule will effectively convert EWS positions into what are currently open category positions. Since the impact of the first two routes are vastly different than the third, it would be best if this subtle aspect of EWS reservation is carefully evaluated and integrated into the implementation of the policy.

If the scope of EWS is expanded but the laws stay silent on this subtle issue, then the outcome of the system will be subject to potential manipulation by politically-motivated authorities who can tilt they system to the benefit of any category they desire. While the difference between the EWS-first VR processing policy and EWS-last VR processing policy are already quite substantial, politically-motivated authorities can also adopt other and less natural policies that nonetheless better serve their own factions. For example, an authority who wants to aid members of OBC can choose an Over-and-Above policy which processes SC and ST prior to EWS, but OBC after EWS. Under this policy the elevated status of caste-based VR protections are only maintained for OBC, but not for SC or ST. Similarly, an authority who wants to disadvantage members of OBC can choose an Over-and-Above policy which processes OBC prior to EWS, but SC, ST after EWS. This policy, in contrast, maintains the elevated status of caste-based VR protections for SC and ST, but not for OBC.

4. Generalized Model with Horizontal Reservations and Extended Analysis

In addition to its primary vertical reservations, India also has horizontal reservations that serve as its secondary affirmative action policy. Since the discussions in the country on its contested EWS reservation and its ongoing case at the Constitution Bench of the Supreme Court are entirely focused on the primary VR policy, in Sections 2 and 3 we also assumed away India's secondary HR policies. In this section we extend our analysis to the general version of the model with both VR and HR policies, and show that our analysis and conclusions persist in this more complex version of the model. The extended analysis we present in this section builds on Sönmez and Yenmez (2022a), and it is valuable not only because it establishes the robustness of our analysis, but also because it provides the nuts and bolts mechanisms policymakers can use to implement their VR and HR policies in the field.[24]

4.1. Horizontal Reservation Policies. In addition to the VR-protected categories in $\mathcal{R}$ that are associated with the primary VR protections, there is a finite set $\mathcal{T}$ of (horizontal) **traits** associated with the secondary HR protections. Each individual $i \in \mathcal{I}$ has a (possibly empty) set of traits, denoted by $\tau_i \in 2^{\mathcal{T}}$. Let $\tau = \left(\tau_i\right)_{i\in\mathcal{I}} \in \left(2^{\mathcal{T}}\right)^{|\mathcal{I}|}$ denote the profile of individual traits. Individuals with these trait are provided with easier access to positions through a second (but less powerful) type of an affirmative action policy.

HR protections are provided within each vertical category of positions (including the open category).[25] For any trait $t \in \mathcal{T}$ and subject to availability of individuals with trait t, priority access is given to individuals with trait t for $q^o_t \in \mathbb{N}$ of the open-category positions. These are referred to as **open-category HR-protected positions for trait t**. Similarly, for any $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$, $c \in \mathcal{R}$ and $t \in \mathcal{T}$, subject to the availability of individuals in $\mathcal{E}^c(\rho)$ with trait t, priority access is given to individuals in $\mathcal{E}^c(\rho)$ with trait t for $q^c_t \in \mathbb{N}$ of the category-c positions. These are referred to as **category-c HR-protected positions for trait t**. Observe that, in contrast to VR protections which are provided on an Over-and-Above basis, HR protections are provided within each vertical category on a "minimum guarantee" basis. This means that positions obtained without invoking the benefits of the HR

[24]The choice rules formulated in Sections 2 and 3, most notably the meritorious O&A, EWS-last O&A, and EWS-first O&A choice rules, are still valuable for real-life implementation in India, because it is fairly common in the country to implement its VR policies first as an initial step, and then makings adjustments as needed to accommodate its secondary HR policies. Indeed, the suggested implementation of the HR policies are typically given in this form in major court rulings on horizontal reservations such as *Saurav Yadav (2020)*. Therefore, the choice rules that are presented in Sections 2 and 3 for the basic model can be thought as allocation mechanisms in the main step prior to any adjustments.

[25]Because of this feature, HR protections are sometimes referred to as *interlocking reservations*.

policy still accommodate the HR protections. In addition, unlike the VR-protected positions which are exclusively set aside for their beneficiaries, the HR-protected positions merely provide priority access for their beneficiaries. It is these two technical aspects which make the HR policy a secondary form of affirmative action policy.

Let $q = \left((q^v)_{v\in\mathcal{V}}, (q^\nu_t)_{(\nu,t)\in\mathcal{V}\times\mathcal{T}}\right)$ denote the vector that specifies (ii) the number of all positions at each vertical category, and (iii) the number of HR-protected positions for each trait and category of positions. We refer to vector q as the **reservation vector**. Throughout this section, we fix the profile of individual traits τ and the reservation vector q.

4.1.1. *The HR Compliance Function.* Fix a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ and a category $v \in \mathcal{V}$. The following technical construction, first formulated in Sönmez and Yenmez (2022a), is useful to formulate a measure of compliance with the HR policy.

Definition 15. Let $\eta^v : 2^{\mathcal{I}} \to \mathbb{N}$ denote a **category-v HR compliance function** that gives the maximum number of category-v HR-protected positions that can be awarded to eligible individuals in $I \bigcap \mathcal{E}^v(\rho)$, for any set of individuals $I \subseteq \mathcal{I}$.

If each individual has at most one trait, the case referred to as **non-overlapping HR protections** in Sönmez and Yenmez (2022a), this function is simply given as follows: For any $I \subseteq \mathcal{I}$,

$$\eta^v(I) = \sum_{t\in\mathcal{T}} \min\left\{|\{i \in I \cap \mathcal{E}^v(\rho) \,:\, t \in \tau_i\}|, q^v_t\right\}.$$

Observe that, for each trait $t \in \mathcal{T}$, the function $\min\left\{|\{i \in I \cap \mathcal{E}^v(\rho) \,:\, t \in \tau_i\}|, q^v_t\right\}$ gives the total number category-v and trait-t HR-protected positions that are honored by the set of individuals I, and therefore, when aggregated across all traits the formula gives the total number of HR-protected positions that are honored by I.

If an individual can have multiple traits, the case referred to as **overlapping HR protections** in Sönmez and Yenmez (2022a), then the formulation of HR compliance function is slightly more involved, and it involves a *maximal assignment of individuals to traits*. Fortunately, as further explained in Appendix A, it is straightforward to calculate the maximum number of HR-protected positions that can be accommodated by any set of individuals $I \subseteq \mathcal{I}$ for any category $v \in \mathcal{V}$ through various computationally efficient maximum cardinality matching algorithms in bipartite graphs.

4.2. Generalized Axioms. We next present extensions of the primary axioms in Section 4.2, and introduce an additional one that formulates the accommodation of HR protections. Throughout this section, fix a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$.

As it is discussed in depth in Sönmez and Yenmez (2022a), each of the following four axioms are mandated throughout India with the Supreme Court judgment *Saurav Yadav (2020)*.[26] Throughout this section, we focus on choice rules that satisfy all four axioms.

The first axiom is the *non-wastefulness*, which is already presented in Definition 4 at Section 4.2.

The second axiom is new, and it simply requires that as many HR-protected positions to be honored as possible at each vertical category of positions. When HR-protected groups are non-overlapping, this simply means not ignoring HR protections. When HR protections are overlapping, it also implies a maximal assignment of individuals to HR-protected positions.

Definition 16. A choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu\in\mathcal{V}}$ satisfies **maximal accommodation of HR protections**, if for any $I \subseteq \mathcal{I}$, $v \in \mathcal{V}$, and $j \in \big(I \cap \mathcal{E}^v(\rho)\big) \setminus \widehat{C}(\rho; I)$,

$$\eta^v\big(C^v(\rho; I)\big) = \eta^v\big(C^v(\rho; I) \cup \{j\}\big).$$

The third axiom requires that no individual receives a position at any category $v \in \mathcal{V}$ at the expense of another eligible individual, unless she either has higher merit-score or awarding her the position increases the number of HR-protected positions that are honored at category v.

Definition 17. A choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu\in\mathcal{V}}$ satisfies **no justified envy** if, for any $I \subseteq \mathcal{I},\ v \in \mathcal{V},\ i \in C^v(\rho; I)$, and $j \in \big(I \cap \mathcal{E}^v(\rho)\big) \setminus \widehat{C}(\rho; I)$,

$$\sigma_i > \sigma_j \quad \text{or} \quad \eta^v\big(C^v(\rho; I)\big) > \eta^v\Big(\big(C^v(\rho; I) \setminus \{i\}\big) \cup \{j\}\Big).$$

The last axiom requires that, an individual who is "deserving" of an open-category position

- either because she has a sufficiently high merit score, or
- because she helps honor a higher number of open-category HR-protected positions,

should be awarded an open-category position and not a position that is VR-protected.

Definition 18. A choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu\in\mathcal{V}}$ satisfies **compliance with VR protections** if, for any $I \subseteq \mathcal{I}$, $c \in \mathcal{R}$, and $i \in C^c(\rho; I)$, we have

[26]Indeed, as it is thoroughly discussed in Sönmez and Yenmez (2022a), the failure of the *no justified envy* axiom under a choice rule that had been mandated in the country between years 1995-2020 resulted countless litigations in the country, and resulted enforcement of this axiom with *Saurav Yadav (2020)*. That is, formulation and enforcement of the *no justified envy* axiom is the primary purpose of this important judgment. The judgment, however, also clarified what it means "to deserve an open-category position on the basis of merit" in the presence of HR protections, and also enforced the axiom *compliance with VR protections*.

(1) $|C^o(\rho; I)| = q^o$,
(2) for each $j \in C^o(\rho; I)$,

$$\sigma_j > \sigma_i \quad \text{or} \quad \eta^o\big(C^o(\rho; I)\big) > \eta^o\Big(\big(C^o(\rho; I) \setminus \{j\}\big) \cup \{i\}\Big), \text{ and}$$

(3) $\eta^o\big(C^o(\rho; I) \cup \{i\}\big) \not> \eta^o\big(C^o(\rho; I)\big)$.

4.3. Sequential Meritorious Horizontal Choice Rules. In this section we generalize the class of sequential choice rules to the more general version of the problem with HR policy. The core "engine" of this class is the *meritorious horizontal* choice rule, a single-category choice rule that is originally introduced in Sönmez and Yenmez (2022a), and which replaces the serial dictatorship to allocate positions at any vertical category.

Throughout this section, we fix a profile $\rho \in \big(2^{\mathcal{R}}\big)^{|\mathcal{I}|}$ of category memberships.

The following auxiliary definition simplifies the formulation of the meritorious horizontal choice rule.

Definition 19. Given a category $v \in \mathcal{V}$ and a set of individuals $I \subseteq \mathcal{E}^v(\rho)$, an individual $i \in \mathcal{E}^v(\rho) \setminus I$ **increases the (category-v) HR utilization of I** if

$$\eta^v(I \cup \{i\}) = \eta^v\big(I\big) + 1.$$

Given a category $v \in \mathcal{V}$ and a set of individuals $I \subseteq \mathcal{I}$, the outcome of the meritorious horizontal choice rule $C^v_{ⓜ}\Big(\rho; .\Big)$ is obtained with the following procedure.

(Category-v) Meritorious Horizontal Choice Rule $C^v_{ⓜ}\Big(\rho; .\Big)$

Step 1.0 (Initiation): Let $I_0 = \emptyset$.

Step 1.k ($k \in \{1, \ldots, \sum_{t \in \mathcal{T}} q^v_t\}$): Assuming such an individual exists, choose the highest merit-score individual in $\big(I \cap \mathcal{E}^v(\rho)\big) \setminus I_{k-1}$ who increases the category-v HR utilization of I_{k-1}. Denote this individual by i_k and let $I_k = I_{k-1} \cup \{i_k\}$. If no such individual exists, proceed to Step 2.

Step 2: For unfilled positions, choose highest merit-score unassigned individuals in $\big(I \cap \mathcal{E}^v(\rho)\big)$ until either all positions are filled or all eligible individuals are selected.

We are ready to formulate the class of *sequential meritorious horizontal (SMH)* choice rules.

Fix an order of precedence $\rhd \in \Delta$. Given a set of individuals $I \subseteq \mathcal{I}$, the outcome of the sequential meritorious horizontal choice rule $C_{ⓢ}(\rhd, \rho; .)$ is obtained with the following procedure.

SMH Choice Rule $C_{ⓢ}(\rhd, \rho; .) = \big(C^{\nu}_{ⓢ}(\rhd, \rho; .)\big)_{\nu \in \mathcal{V}}$

Step 0 (Initiation): Let $I_0 = \emptyset$.
Step k ($\boldsymbol{k \in \{1, \ldots, |\mathcal{V}|\}}$): Let v_k be the category which has the k^{th} highest order of precedence under $\rhd$.

$$C_{Ⓢ}^{v_k}(\rhd, \rho; I) = C_{ⓜ}^{v_k}\Big(\rho; \big(I \setminus I_{k-1}\big) \cap \mathcal{E}^{v_k}(\rho)\Big)$$

Let $I_k = I_{k-1} \cup C_{Ⓢ}^{v_k}(\rhd, \rho; I)$.

Under this procedure the meritorious horizontal choice rule is applied sequentially for each vertical category $v \in \mathcal{V}$, following their order of precedence under $\rhd$.

4.3.1. *Preliminary Results on SMH Choice Rules.* The class of SMH choice rules generalizes the *two-step meritorious horizontal (2SMH)* choice rule $C_{Ⓜ}^{2s}(\rho; .) = \big(C_{Ⓜ}^{2s,\nu}(\rho; .)\big)_{\nu \in \mathcal{V}}$, originally introduced by Sönmez and Yenmez (2022a). 2SMH choice rule $C_{Ⓜ}^{2s}(\rho; .)$ is defined for environments with non-overlapping VR protections only, and in these environments it is equivalent to any SMH that where the open category has the highest order of precedence.[27] That is,

$$C_{Ⓜ}^{2s}(\rho; .) = C_{Ⓢ}(\rhd^o, \rho; .) \qquad \text{for any } \rhd^o \in \Delta^o.$$

As such, the open-category is processed prior to any VR-protected category under the choice rule 2SMH. Just as the relative processing sequence of VR-protected categories are immaterial under the O&A choice rule when they are non-overlapping, the same is also true for the 2SMH choice rule.

The following characterization by Sönmez and Yenmez (2022a) generalizes Proposition 0, thus establishing that the 2SMH choice rule assumes the central role of the O&A choice rule in the presence of HR policy.

Theorem 0. *Fix a profile of category memberships $\rho = (\rho_i)_{i \in \mathcal{I}} \in \big(2^{\mathcal{R}}\big)^{|\mathcal{I}|}$ such that VR-protected categories are non-overlapping. Then, a choice rule $C(\rho; .)$ satisfies non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections if and only if $C(\rho; .) = C_{Ⓜ}^{2s}(\rho; .)$.*

Since VR-protected categories are currently non-overlapping in India, the legislation in India has airtight implications by Theorem 0. Indeed, focusing on a simpler version of problem with non-overlapping HR protections, the 2SMH choice rule was recently endorsed in the country by the Supreme Court judgment *Saurav Yadav (2020)*, and enforced in the state of Gujarat by the high court judgment *Tamannaben Ashokbhai Desai (2020)*.[28]

[27] Since the outcome of the 2SMH choice rule is independent of the choice of an order of precedence in Δ^o, the parameters is suppressed in $C_{Ⓜ}^{2s}(\rho; .)$.

[28] When HR protections are non-overlapping, Sönmez and Yenmez (2022a) refers to the resulting simpler version of the 2SMH choice rule as the *two-step minimum guarantee (2SMG) choice rule*.

Since the scope of the EWS category may soon be expanded in the country, some other elements of the class of SMH choice rules, especially those under EWS-first and EWS-last VR processing policies, are of particular interest.

Extending Lemmas 1 and 2, the next two lemmas show that each SMS choice rule satisfies the first three axioms mandated by *Saurav Yadav (2020)*, and for as long as open-category has higher order of precedence than any other VR-protected category, it also satisfies the fourth axiom.

Lemma 5. *For any* $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *and* $\rhd \in \Delta$*, the SMH choice rule* $C_{Ⓢ}(\rhd, \rho; .)$ *satisfies non-wastefulness, maximal accommodation of HR protections, and no justified envy.*

Lemma 6. *For any* $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *and* $\rhd \in \Delta^o$*, the SMH choice rule* $C_{Ⓢ}(\rhd, \rho; .)$ *satisfies compliance with VR protections.*

Extending Lemma 3, the next lemma further shows that, even if non-regular VR policies are adopted at some (or all) of potentially overlapping VR-protected categories, any choice rule that satisfies the Supreme Court's axioms still has to allocate the open-category positions to the same individuals who would have received them under the 2SMH choice rule. Thus, any deviation from the outcome of the 2SMH is due to the assignment of VR-protected positions.[29]

Lemma 7. *Fix a profile of category memberships* $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$*, and another profile of category memberships* $\rho' \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ *that is non-overlapping. Let* $C(\rho; .) = \left(C^{\nu}(\rho; .)\right)_{\nu \in \mathcal{V}}$ *be any choice rule that satisfies non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections. Then, for any set of individuals* $I \subseteq \mathcal{I}$*,*

$$C^{o}(\rho; I) = C_{Ⓜ}^{2s,o}(\rho'; I).$$

4.3.2. *Extended Results on EWS-last VR Processing Policy.* In this section we follow the notation and terminology used in Section 3.1, and directly extend the results in this section for the most general version of the problem with horizontal reservations. Below, Lemma 8 extends Lemma 4, Proposition 3 extends Proposition 2, Theorem 4 extends Theorem 2, and Corollary 2 extends Corollary 1 respectively. In all these results, the EWS-last SMH choice rule plays is the same role the EWS-last sequential choice rule plays in Section 3.1, and the role 2SMH choice rule plays is the same role the role O&A choice rule in Section 3.1.

[29]While this result is a slightly stronger version of Lemma 10 in Sönmez and Yenmez (2022a) (due to overlapping VR-protected categories and different profiles of category memberships used under the two choice rules), the changes are superfluous and the two proofs are analogous. Hence, we omit the proof of Lemma 7.

Lemma 8. *Fix the profile of category memberships as $\mathring{\rho}$. For any set of individuals $I \subseteq \mathcal{I}$, the maximal set of individuals who suffer from a violation of the Equality Code under the choice rule $C^{2s}_{Ⓜ}(\mathring{\rho};.)$ for I is uniquely defined.*

Proposition 3. *Fix the profile of category memberships as ρ^* and the order of precedence as $\trianglerighteq \in \Delta^o_e$. Then, the EWS-last SMH choice rule $C_{Ⓢ}(\trianglerighteq, \rho^*;.)$ abides by the Equality Code and it satisfies non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections.*

Theorem 4. *Consider any set of individuals $I \subseteq \mathcal{I}$. Then, the set of individuals*

$$C_{Ⓢ}(\trianglerighteq, \rho^*; I) \setminus C^{2s}_{Ⓜ}(\mathring{\rho}; I)$$

is equal to the maximal set of individuals who suffer from a violation of the Equality Code under the choice rule $C^{2s}_{Ⓜ}(\mathring{\rho};.)$ for I.

Corollary 2. *Consider any set of individuals $I \subseteq \mathcal{I}$. Then, we have*

$$C_{Ⓢ}(\trianglerighteq, \rho^*; I) = C^{2s}_{Ⓜ}(\mathring{\rho}; I)$$

if and only if the set of individuals I are materially unaffected by the violation of the Equality Code under the choice rule $C^{2s}_{Ⓜ}(\mathring{\rho};.)$.

4.3.3. *Extended Results on EWS-first VR Processing Policy.* In this section we follow the notation and terminology used in Section 3.2, and directly extend the main characterization in this section for the EWS-first VR processing policy for the most general version of the problem with horizontal reservations.

In order to do this, we first have to extend the axiom which expands the elevated status of caste-based VR protections to the general version of the problem with HR policy. Axiom 20 below states that, a member of a caste-based VR category who is also eligible for EWS should not use up a position for her caste-based VR category if she merits an EWS position either due to her merit score or because she increases the HR utilization at EWS category.

Definition 20. A choice rule $C(\rho;.) = \big(C^{\nu}(\rho;.)\big)_{\nu \in \mathcal{V}}$ **maintains the elevated status of caste-based VR protections** if, for any $I \subseteq \mathcal{I}$, $c \in \mathcal{R}^0$, and $i \in C^c(\rho; I) \bigcap \mathcal{E}^e(\rho)$,

(1) $|C^e(\rho; I)| = q^e$,
(2) for each $j \in C^e(\rho; I)$,

$$\sigma_j > \sigma_i \quad \text{or} \quad \eta^e\big(C^e(\rho; I)\big) > \eta^e\Big(\big(C^e(\rho; I) \setminus \{j\}\big) \cup \{i\}\Big), \text{ and}$$

(3) $\eta^e\big(C^e(\rho; I) \cup \{i\}\big) \not> \eta^e\big(C^e(\rho; I)\big)$.

Our final result states that, EWS-first SMH choice rule is the only rule that satisfies this additional axiom along with the mandates of *Saurav Yadav (2020)*.

Theorem 5. *Fix a profile of category memberships $\rho = (\rho_i)_{i\in\mathcal{I}} \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ such that, for any $i \in \mathcal{I}$,*

$$|\rho(i)| \leq 2 \quad \text{and} \quad |\rho(i)| = 2 \implies e \in \rho(i).$$

Fix an order of precedence as $\overline{\rhd} \in \Delta^{o,e}$. Then, a choice rule $C(\rho;.)$ maintains the elevated status of caste-based VR protections and it satisfies non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections if and only if $C(\rho;.) = C_{Ⓢ}(\overline{\rhd}, \rho;.)$.

4.4. Implementation of Simultaneous VR Processing Policy with HR Protections. We conclude this section by formulating a choice rule that processes all VR-protected categories simultaneously, in the presence of both VR and HR policies.

Fix a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$.

Definition 21. The **VR-protected HR-maximality function** $\eta^{\mathcal{R}} : 2^{\mathcal{I}} \to \mathbb{N}$ is defined as for any $I \subseteq \mathcal{I}$, $\eta^{\mathcal{R}}(I)$ is the maximum number of HR-protected positions in VR-protected categories that can be accommodated for individuals in I.

Definition 22. Consider the profile of category memberships ρ. For any set of individuals $I \subseteq \mathcal{I}$ and individual $i \in \mathcal{I} \setminus I$, **individual i increases the overall VR-protected HR-utilization of I** if,

$$\eta^{\mathcal{R}}(I \cup \{i\}) = \eta^{\mathcal{R}}(I) + 1.$$

Fix $m \in \mathbb{R}_+$ such that

$$m > \max_{i\in\mathcal{I}} \sigma_i.$$

Given a set of individuals $I \subseteq \mathcal{I}$, we find the outcome of the adaptive transversal (AT) choice rule in two steps:

Adaptive Transversal Choice Rule $C_{Ⓐ}(\rho;.) = \left(C^{\nu}_{Ⓐ}(\rho;.)\right)_{\nu\in\mathcal{V}}$:

Step 1. Open category is processed using the single-category meritorious horizontal rule:

$$C^{o}_{Ⓐ}(\rho; I) = C^{o}_{ⓜ}(\rho; I).$$

Step 2. All VR-protected categories are processed in parallel in two substeps. Let $J = I \setminus C^{o}_{Ⓐ}(\rho; I)$ be the set of remaining individuals in I after Step 1.

Step 2a. We determine a set of individuals $K \subseteq J$ who benefit from HR-protected positions of VR-protected categories:

Step 2a.0 (Initiation): Let $J_0 = \emptyset$.

Step 2a.k ($\boldsymbol{k \in \{1, \ldots, \sum_{c\in\mathcal{R}} q^c + 1\}}$): Assuming such an individual exists, choose the highest merit-score individual in $J \setminus J_{k-1}$ who increases the overall VR-protected HR-utilization of J_{k-1} . Denote this individual by j_k and let $J_k = J_{k-1} \cup \{j_k\}$. If no such individual exists, then end the process by setting $K = J_{k-1}$ and continue with Step 2b.

Step 2b. We determine a set of individuals $M \subseteq J$ such that $M \supseteq K$ as the benefactors of VR protections in the choice rule $C_{\text{AT}}(\rho, .)$ for I:

Form an auxiliary merit score vector $\tilde{\sigma}$ such that for

(1) each $i \in K$, $\tilde{\sigma}_i = \sigma_i + m$, and

(2) for each $i \in \mathcal{I} \setminus K$, $\tilde{\sigma}_i = \sigma_i$.

Step 2b.0 (Initiation): Let $J_0 = \emptyset$.

Step 2b.k ($k \in \{1, \ldots, \sum_{c \in \mathcal{R}} q^c + 1\}$): Assuming such an individual exists, choose the highest $\tilde{\sigma}$-score individual in $J \setminus J_{k-1}$ who increases the VR-utilization for J_{k-1}. Denote this individual by j_k and let $J_k = J_{k-1} \cup \{j_k\}$. If no such individual exists, then end the process by setting $M = J_{k-1}$.

Assign a disjoint subset of eligible individuals in M to each category $c \in \mathcal{C}$ so that[30]

$$\cup_{c \in \mathcal{R}} C^c_{\text{AT}}(\rho; I) = M.$$

Remark 2. The AT choice rule $C_{\text{AT}}(\rho; .)$ is a generalization of both 2SMH choice rule $C^{2s}_{\text{M}}(\rho; .)$ and mO&A choice rule $C_{\text{OA}}(\rho; .)$:

(1) If category memberships are not overlapping in ρ, then for any $I \subseteq \mathcal{I}$,

$$C_{\text{AT}}(\rho; I) = C^{2s}_{\text{M}}(\rho; I).$$

(2) If there are no HR-protected positions, then for any $I \subseteq \mathcal{I}$,

$$\widehat{C}_{\text{AT}}(\rho; I) = \widehat{C}_{\text{OA}}(\rho; I).$$

5. Conclusion

India has a constitutionally-protected affirmative action system that involves a very complex set of normative goals and requirements. While Supreme Court and state high court justices in the country have historically done an exemplary job of rigorously formulating these normative principles and providing guidance on their implementation, due

[30] In Appendix A, we explain exactly how we construct each component $C^c(\rho; I)$. There is a benign multiplicity in this definition, possibly different component set profiles $\left(C^c_{\text{AT}}(\rho; I)\right)_{c \in \mathcal{R}}$ can be constructed so that each of them choose a distinct eligible subset of individuals in M and their union is exactly M. This technically makes the adaptive transversal choice rule a correspondence. However, its aggregate choice rule $\widehat{C}_{\text{AT}}(\rho; .)$ is uniquely defined.

to the sheer complexity of the problem, in some cases they failed to identify the collective implications of these principles or how changes in various aspects of the the applications may interfere with them Sönmez and Yenmez (2022a,b). These challenges in turn resulted in inconsistencies between judgments, loopholes in the system, and various unintended consequences. As emphasized in Li (2017),

> In addition to studying cause and effect in markets, economists also have a comparative advantage in stating precisely the normatively-relevant properties of complex systems [...]

The primary aim of this paper is taking advantage of this comparative advantage by presenting how an intuitive compromise policy brought recently to the hearings of a major Supreme Court case on a potential repeal of the 103rd Amendment of the Constitution can both

- result in a major loophole in the system if adopted by the court without any additional structure,
- but at the same time can be further refined to three policies that serve three distinct (but each reasonable) normative objectives.

The challenge before the court resembles a policy decision in Boston Public Schools that de facto eliminated the role of its walk-zone policies between years 1999 and 2013 in assigning children to public schools. After discovering that their policy was superfluous and misleading, the walk zone policy was abandoned by the city altogether. As reported in Appendix D of Dur et al. (2018), during a March 2013 speech to Boston School Committee, Superintended Carol Johnson justified this decision as follows:

> After viewing the final MIT and BC presentations on the way the walk zone priority actually works, it seems to me that it would be unwise to add a second priority to the Home-Based model by allowing the walk zone priority be carried over. [. . .] Leaving the walk zone priority to continue as it currently operates is not a good option. We know from research that it does not make a significant difference the way it is applied today: although people may have thought that it did, the walk zone priority does not in fact actually help students attend schools closer to home. The External Advisory Committee suggested taking this important issue up in two years, but I believe we are ready to take this step now. We must ensure the Home-Based system works in an honest and transparent way from the very beginning.

Our message in this paper is similar to that of Superintendent Johnson, albeit for an application with a much larger scope and scale. If India extends the scope of the EWS reservation, it is important to ensure that the new policy is implemented as intended in a transparent way.

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Appendices

Appendix A. Preliminaries

A.1. Matchings, Matroids, Greedy Algorithm, and Choice Rules. In this subsection, we introduce auxiliary mathematical structures that will be useful in formalizing some of our definitions and proving our results.

Fix a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$. Let $\mathcal{E}^c = \mathcal{E}^c(\rho)$ for each $c \in \mathcal{R}$.

Consider a reservation vector $q = \left(q^\nu, (q^\nu_t)_{t\in\mathcal{T}}\right)_{\nu\in\mathcal{V}}$ so that there are potentially HR-protected positions in each category. Let $\mathcal{Q}$ be the set of reservation vectors.

Define for each $v \in \mathcal{V}$,

$$q^v_{t_\emptyset} = q^v - \sum_{t\in\mathcal{T}} q^v_t$$

where $t_\emptyset$ refers to **nil trait**, which we use to denote a position of any category v that is not protected by horizontal reservations.

Definition A.1. Let $I \subseteq \mathcal{I}$. A **matching** is a mapping $\mu : I \to \left(\mathcal{V} \times (\mathcal{T} \cup \{t_\emptyset\})\right) \cup \{\emptyset\}$ such that

- for each $i \in I$, $\mu(i) = \emptyset$ or $\mu(i) = (v,t)$ with $i \in \mathcal{E}^v$ and $t \in \tau_i \cup \{t_\emptyset\}$, and
- for each $v \in \mathcal{V}$ and $t \in \mathcal{T} \cup \{t_\emptyset\}$, $|\mu^{-1}(v,t)| \leq q^v_t$.

For an individual i, $\mu(i) = \emptyset$ refers to her remaining **unmatched**. Let $\mathcal{M}(q, I)$ be the set of matchings for I under q.[31]

The set of individuals matched by a matching $\mu \in \mathcal{M}(q,I)$ with a category $v \in \mathcal{V}$ is denoted as

$$I_{\mu,v} = \Big\{ i \in I : \mu(i) \in \{v\} \times \big(\mathcal{T} \cup \{t_\emptyset\}\big) \Big\},$$

and the overall set of individuals matched by μ is denoted as

$$I_\mu = \cup_{\nu \in \mathcal{V}} I_{\mu,\nu}.$$

Using the auxiliary concept of matchings, we define the following three functions, which were formally defined for special cases in the main text:

Definition A.2. For each $v \in \mathcal{V}$, **the category-v HR-compliance function** $\eta^v : 2^{\mathcal{I}} \to \mathbb{N}$ is defined for any $I \subseteq \mathcal{I}$ as

$$\eta^v(I) = \max_{\mu \in \mathcal{M}(q,I)} \Big| \{ i \in I : \mu(i) \in \{v\} \times \mathcal{T} \} \Big|.$$

Definition A.3. The **VR-protected HR-maximality function** $\eta^{\mathcal{R}} : 2^{\mathcal{I}} \to \mathbb{N}$ is defined for any $I \subseteq \mathcal{I}$ as

$$\eta^{\mathcal{R}}(I) = \max_{\mu \in \mathcal{M}(q,I)} \Big| \{ i \in I : \mu(i) \in \mathcal{R} \times \mathcal{T} \} \Big|.$$

Definition A.4. The **VR-maximality function** $\beta : 2^{\mathcal{I}} \to \mathbb{N}$ is defined for any $I \subseteq \mathcal{I}$ as

$$\beta(I) = \max_{\mu \in \mathcal{M}(\tilde{q},I)} \Big| \{ i \in I : \mu(i) \in \mathcal{R} \times \{t_\emptyset\} \} \Big|.$$

where auxiliary reservation vector $\tilde{q} \in \mathcal{Q}$ is equal to reservation vector q in every component except for trait positions in VR-protected categories such that $\tilde{q}^c_t = 0$ for each $c \in \mathcal{R}$.

The auxiliary reservation vector $\tilde{q}$ in Definition A.4 effectively converts all positions of a VR-protected category to nil-trait positions. Therefore, $\beta(I)$ is the maximum number of individuals in I that can be accommodated by VR-protected positions regardless of traits of the individuals.

The following mathematical structure is useful to understand properties of our choice rules.

Definition A.5. A pair of finite sets $(E, \mathcal{X})$ is a **matroid** if $\mathcal{X} \subseteq 2^E$ and

[31] Although we do not change the real reservation vector q after it is set, we keep a reservation vector in the argument of the set of matchings. This is due to the fact that we introduce auxiliary reservation vectors by modifying q, which play an important role in formal definitions and proofs.

(1) if $F \in \mathcal{X}$, then for any $G \subsetneq F$ we have $G \in \mathcal{X}$ (thus, $\emptyset \in \mathcal{X}$).
(2) if $F, G \in \mathcal{X}$ such that $|G| < |F|$, then there is an element $x \in F \setminus G$ such that $G \cup \{x\} \in \mathcal{X}$.

Each element of $\mathcal{X}$ is called an **independent set** of matroid $(E, \mathcal{X})$.

Next we define a specific matroid for our purposes.

For any set of individuals $I \subseteq \mathcal{I}$, define a collection of subsets of I

$$\boldsymbol{\mathcal{A}(q, I)} = \{J \subseteq I : \ \exists\, \mu \in \mathcal{M}(q, I) \text{ s.t. } J \subseteq I_\mu\}. \tag{1}$$

We refer to each set $J \in \mathcal{A}(q, I)$ as an **assignable subset of $\boldsymbol{I}$ under $\boldsymbol{q}$.**

Lemma A.1 (Edmonds and Fulkerson (1965))**.** *Consider the reservation vector q. For any set of individuals $I \subseteq \mathcal{I}$, the pair $\big(I, \mathcal{A}(q, I)\big)$ is a matroid.*

Matroid $\big(I, \mathcal{A}(q, I)\big)$ is called a **transversal matroid** (e.g., see Lawler, 2001). Observe that assignable subsets of I under q are exactly the independent sets of transversal matroid $\big(I, \mathcal{A}(q, I)\big)$.

We also consider auxiliary merit score vectors obtained from the original merit score vector. Let $\Sigma \subsetneq \mathbb{R}_+^{|\mathcal{I}|}$ be the set of merit score vectors, which are non-negative real valued and no two individuals have the same score. Given a merit score vector $\sigma \in \Sigma$, we find a particular assignable subset of I under q that we denote as $G(q, I; \sigma)$ through an iterative procedure.[32]

Greedy algorithm for transversal matroid $\big(I, \mathcal{A}(q, I)\big)$ under merit score vector σ:

Step 0 (Initiation): Let $I_0 = \emptyset$.
Step k $\big(\boldsymbol{k \in \{1, \ldots, |I| + 1\}}\big)$: Assuming such an individual exists, choose the highest σ-score individual whom we denote as $i_k \in I \setminus I_{k-1}$ such that $I_{k-1} \cup \{i_k\} \in \mathcal{A}(q, I)$ and let $I_k = I_{k-1} \cup \{i_k\}$. If no such individual exists, then end the process by setting $\boldsymbol{G(q, I; \sigma)} = I_{k-1}$.

We refer to the outcome set $G(q, I; \sigma)$ of this procedure as **the greedily assignable subset in $\boldsymbol{\mathcal{A}(q, I)}$ under $\boldsymbol{\sigma}$.**

While the greedily assignable subset is uniquely defined, there can be multiple matchings that assign the same set of individuals to different category – trait/nil-trait pairs. We refer to any matching $\mu \in \mathcal{M}(q, I)$ such that

$$I_\mu = G(q, I; \sigma)$$

[32]The greedy algorithm is defined for general matroids, although we define it using special wording for a transversal matroid. Lemma A.2 also holds for the greedy algorithm for general matroids.

as **a greedy matching in $\boldsymbol{\mathcal{M}(q, I)}$ under $\boldsymbol{\sigma}$**. The following important property of the greedy assignable subset is key in proving Theorem 1 .

Lemma A.2 (Gale (1968))**.** *For any reservation vector $q \in \mathcal{Q}$, merit score vector $\sigma \in \Sigma$, set of individuals $I \subseteq \mathcal{I}$, and assignable subset $J \in \mathcal{A}(q, I)$, the greedily assignable subset $G(q, I; \sigma)$ Gale dominates assignable subset J.*

An immediate corollary to this result is as follows using the definition of Gale domination.

Corollary 3. *For any reservation vector $q \in \mathcal{Q}$, merit score vector $\sigma \in \Sigma$, set of individuals $I \subseteq \mathcal{I}$, and assignable subset $J \in \mathcal{A}(q, I)$, the greedily assignable subset $G(q, I; \sigma)$ satisfies $|G(q, I; \sigma)| \geq |J|$.*

Greedy algorithm has a special place in our choice rule constructions.

Remark 3 (Sönmez and Yenmez (2022a))**.** Let $v \in \mathcal{V}$. The category-v meritorious horizontal choice rule utilizes the greedy algorithm for any $I \subseteq \mathcal{I}$ for transversal matroid $\big(I, \mathcal{A}(q^*, I)\big)$ under σ in Step 1 of its procedure where q^* is obtained from q by setting (i) for each $\nu \in \mathcal{V} \setminus \{v\}$, $q^{*\nu} = 0$ and (ii) $q^{*v} = q^v$ and $q^{*v}_t = q^v_t$ for each $t \in \mathcal{T}$.

Remark 4. Assuming there are no HR-protected positions in q, the mO&A choice rule $C_{\text{OA}}(\rho; .)$ utlizes the greedy algorithm for any $I \subseteq \mathcal{I}$ for transversal matroid $\big(J, \mathcal{A}(q', J)\big)$ under σ in Step 2 of its procedure where

- $J = I \setminus C^o_{\text{OA}}(\rho; I)$, and
- q' is obtained from q by setting (i) $q'^o = 0$ and (ii) $q'^c = q^c$ for each $c \in \mathcal{R}$.

We pick a greedy matching $\mu' \in \mathcal{M}(q', J)$ under σ to determine the VR-protected category components of $C_{\text{OA}}(\rho; .)$. For each $c \in \mathcal{R}$, we set

$$C^c_{\text{OA}}(\rho; I) = I_{\mu', c}.$$

Thus,

$$\cup_{c \in \mathcal{R}} C^c_{\text{OA}}(\rho; I) = G(q', J; \sigma).$$

Remark 5. The adaptive transitive choice rule $C_{\text{AT}}(\rho; .)$ utilizes the greedy algorithm for any $I \subseteq \mathcal{I}$ in Step 2 of its procedure as follows:

(1) Step 2a is equivalent to the greedy algorithm for the transversal matroid $\big(J; \mathcal{A}(\overline{q}, J)\big)$ under σ where
- $J = I \setminus C^o_{\text{AT}}(\rho; I)$, and
- $\overline{q}$ is obtained from q by setting (i) $\overline{q}^o = 0$ and (ii) for each $c \in \mathcal{R}$, $\overline{q}^c = \sum_{t \in \mathcal{T}} q^c_t$ and $\overline{q}^c_t = q^c_t$ for each $t \in \mathcal{T}$.

The outcome set of Step 2a is

$$K = G(\overline{q}, J; \sigma).$$

(2) Step 2b is equivalent to the greedy algorithm for the transversal matroid $\big(J; \mathcal{A}(\tilde{q}, J)\big)$ under $\tilde{\sigma}$ where
- $\tilde{\sigma}$ is obtained from σ by setting (i) for each $i \in K$, $\tilde{\sigma}_i = \sigma_i + m$ and (ii) for each $i \in \mathcal{I} \setminus K$, $\tilde{\sigma}_i = \sigma_i$, and
- $\tilde{q}$ is obtained from q by setting (i) $\tilde{q}^o = 0$ and (ii) for each $c \in \mathcal{C}$, $\tilde{q}^c = q^c$ and $\tilde{q}^c_t = 0$ for each $t \in \mathcal{T}$.

We pick a greedy matching $\tilde{\mu} \in \mathcal{M}(\tilde{q}, J)$ under $\tilde{\sigma}$ to determine the VR-protected category components of $C_{\text{Ⓐⓣ}}(\rho; .)$. For each $c \in \mathcal{R}$,

$$C^c_{\text{Ⓐⓣ}}(\rho; I) = I_{\tilde{\mu}, c}.$$

A.2. Some Properties of Choice Rules. We use the following property in our proofs.

Definition A.6 (Aygün and Sönmez (2013))**.** For any $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$, a choice rule $C(\rho; .)$ satisfies **irrelevance of rejected individuals** if, for any $I \subseteq \mathcal{I}$ and $i \in I \setminus \widehat{C}(\rho; I)$,

$$\widehat{C}\big(\rho; I \setminus \{i\}\big) = \widehat{C}(\rho; I).$$

Lemma A.3 (Sönmez and Yenmez (2022b))**.** *For any $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ and $v \in \mathcal{V}$, the category-v meritorious horizontal choice rule $C^v_{ⓜ}(\rho; .)$ satisfies irrelevance of rejected individuals.*

We also explicitly state an implication of Theorem 0 for single-category choice rules that we use in our proofs.[33]

Lemma A.4 (Sönmez and Yenmez (2022a))**.** *For any $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ and $v \in \mathcal{V}$, a single-category choice rule $C^v(\rho; .)$ maximally accommodates HR protections, satisfies no justified envy, and is non-wasteful if and only if $C^v(\rho; .) = C^v_{ⓜ}(\rho; .)$.*

Appendix B. Proofs

Several results in Section 2 and Section 3 follow from their generalizations in Section 4. In Section 2, Lemma 1 follows from Lemma 5, Lemma 2 follows from Lemma 6 and Lemma 3 follows from Lemma 7 (see Footnote 29). In Section 3, Lemma4 follows from Lemma 8, Proposition 2 follows from Proposition 3, Theorem 2 follows from Theorem 4, and Theorem 3 follows from Theorem 5.

[33] Also see Theorem 2 in Sönmez and Yenmez (2022a).

B.1. Proofs of Results in Section 2.4.

Proof of Proposition 1. Consider the reservation vector q such that $q^v_t = 0$ for each $v \in \mathcal{V}$ and $t \in \mathcal{T}$. Fix a profile of category memberships $\rho \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ and a set of individuals $I \subseteq \mathcal{I}$.

Non-wastefulness: Suppose $v \in \mathcal{V}$ be such that $|C^v_{OA}(\rho; I)| < q^v$, and there exists some $j \in I \setminus \widehat{C}_{OA}(\rho; I)$. If $v = o$, then this contradicts j not being selected in Step 1 of the mO&A choice rule. Suppose $v \in \mathcal{R}$. Thus, even though Step 2 of the procedure leaves vacant positions in category v, j does not receive a position, meaning that she cannot increase the VR-utilization of $\cup_{c \in \mathcal{C}} C^c_{OA}(\rho; I)$. Then $j \notin \mathcal{E}^v$. We showed $C_{OA}(\rho; .)$ is non-wasteful.

No justified envy: Let $v \in \mathcal{V}$ and $j \in I \setminus \widehat{C}_{OA}(\rho; I)$. We have two cases:

(1) Consider the case $v = o$. By construction of Step 1 of the mO&A choice rule, if $i \in C^o_{OA}(\rho; I) = C^o_{sd}(\rho; I)$, then i is chosen instead of j in the category-o serial dictatorship. Hence, $\sigma_i > \sigma_j$.

(2) Consider the case $v \in \mathcal{R}$. Suppose $i \in C^v_{OA}(\rho; I)$. Then $i \in \mathcal{E}^v$. Let $J = I \setminus C^o_{OA}(\rho; I)$. Observe that Step 2 of the mO&A choice rule is the greedy algorithm for matroid $\big(J, \mathcal{A}(q', J)\big)$ where the auxiliary reservation vector q' is obtained from q by setting the open category positions to zero and keeping every other category's positions the same as in q (see Remark 4). Hence, in Step 2, the greedily assignable subset $G(q', J; \sigma)$ is the set of individuals chosen. In particular, $i \in G(\overline{q}, J; \sigma)$.

Suppose, for a contradiction, $\sigma_j > \sigma_i$ and $j \in \mathcal{E}^v$.

We first show that set K defined as $K = \big(G(q', J; \sigma) \setminus \{i\}\big) \cup \{j\}$ is an assignable subset of J under q': To see this, pick a greedy matching $\mu \in \mathcal{M}(q', J)$ assigning i to v. Such a greedy matching exists as $i \in C^v_{OA}(\rho; I)$ (see Remark 4). By definition of a greedy matching, $J_\mu = G(\overline{q}, J; \sigma)$. Since both $i, j \in \mathcal{E}^v$ and there are no HR-protected positions in q', we can modify μ by assigning j to v instead of i and keeping all other assignments the same. Thus, we obtain a matching in $\mathcal{M}(q', J)$ that matches all individuals in K, showing that $K \in \mathcal{A}(q', J)$.

Since $\sigma_j > \sigma_i$, j is processed before i in the process of the greedy algorithm. Then j should be chosen instead of i, as K is an assignable subset of J under q', which includes j and all previously committed individuals in the greedy algorithm prior to j. This is a contradiction to $j \notin G(q', J; \sigma) \subseteq \widehat{C}_{OA}(\rho; I)$. Hence, $\sigma_i > \sigma_j$ or $j \notin \mathcal{E}^v$.

Thus, $C_{OA}(\rho; .)$ satisfies no justified envy.

Compliance with VR protections: Let $c \in \mathcal{R}$ and $i \in C^c_{OA}(\rho; I)$. In executing the procedure of the mO&A choice rule, the open category is processed first and individual i is still available when the VR-protected categories are about to be processed in Step 2. Thus, it should be the case that $|C^o_{\text{OA}}(\rho; I)| = q^o$, as otherwise i would have received a category-o position instead of a category-c position. Moreover, $C^o_{\text{OA}}(\rho; I)$, which is formed by a serial dictatorship, includes the highest σ-score q^o individuals from I, implying that for every $j \in C^o_{\text{OA}}(\rho; I)$, $\sigma_j > \sigma_i$. Thus, $C_{\text{OA}}(\rho; .)$ satisfies compliance with VR protections. ■

Proof of Theorem 1. Consider the reservation vector q such that $q^v_t = 0$ for each $v \in \mathcal{V}$ and $t \in \mathcal{T}$. Fix $I \subseteq \mathcal{I}$. By Lemma 7, for any non-overlapping profile of category memberships $\rho' \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ we have $C^o_{\text{OA}}(\rho; I) = C^{2s,o}_{\text{M}}(\rho'; I)$ and $C^o(\rho; I) = C^{2s,o}_{\text{M}}(\rho'; I)$, implying that

$$C^o_{\text{OA}}(\rho; I) = C^o(\rho; I). \tag{2}$$

Let $J = I \setminus C^o_{\text{OA}}(\rho; I)$. Define the auxiliary reservation vector $q' \in \mathcal{Q}$ such that the open category has no position and each VR-protected category has the same number of positions as it has under q: $q'^o = 0$ and $q'^c = q^c$ for each $c \in \mathcal{R}$. Construct the matching $\mu \in \mathcal{M}(q', J)$ by assigning nil-trait positions of each category $c \in \mathcal{R}$ under μ to the individuals in $C^c(\rho; I)$. Thus, the set $\cup_{c\in\mathcal{R}} C^c(\rho; I)$ is an assignable subset of J under $\overline{q}$, i.e.,

$$\cup_{c\in\mathcal{R}} C^c(\rho; I) \in \mathcal{A}(q', J). \tag{3}$$

Step 2 of the mO&A rule is the greedy algorithm for matroid $\left(J, \mathcal{A}(q', J)\right)$ under σ (see 4), implying that

$$\cup_{c\in\mathcal{C}} C^c_{\text{OA}}(\rho; I) = G(q', J; \sigma).$$

Thus, by Lemma A.2 and Eq. (3), $\cup_{c\in\mathcal{C}} C^c_{\text{OA}}(\rho; I)$ Gale dominates $\cup_{c\in\mathcal{R}} C^c(\rho; I)$. This statement and Eq. (2) show that set $\widehat{C}_{\text{OA}}(q; I)$ Gale dominates set $\widehat{C}(q; I)$. ■

B.2. Proofs of Preliminary Results in Section 4.3.1.

Proof of Lemma 5. Fix $\rho \in (2^{\mathcal{R}})^{|\mathcal{I}|}$ and $\rhd \in \Delta$. Let $\mathcal{E}^c = \mathcal{E}^c(\rho)$ for each $c \in \mathcal{R}$. Fix also $I \subseteq \mathcal{I}$.

Non-wastefulness: Suppose $v \in \mathcal{V}$ be such that $|C^v_{\text{S}}(\rhd, \rho; I)| < q^v$, and there exists some $j \in I \setminus \widehat{C}_{\text{S}}(\rhd, \rho; I)$. Then, just before v is processed in the sequence $\rhd$, j is still available. Moreover, she does not receive a category-v position. Thus, even though Step 2 of the procedure of $C^v_{\text{m}}(\rho; .)$ leaves some vacant jobs in category v, j does not receive a position. Thus, $j \notin \mathcal{E}^v$. Since by definition

$$C^v_{\text{S}}(\rhd, \rho; I) = C^v_{\text{m}}\left(\rho; I \setminus \cup_{\nu\in\mathcal{V}:\nu\lhd v} C^\nu_{\text{S}}(\rhd, \rho; I)\right) \tag{4}$$

and the argument in previous sentence is true for each category v, the SMH choice rule is non-wasteful.

Maximal accommodation of HR protections: Suppose $v \in \mathcal{V}$ and $j \in \big(I \cap \mathcal{E}^v\big) \setminus \widehat{C}_{\circledS}(\rhd, \rho; I)$. In processing the sequence $\rhd$ in executing the SMH choice rule for I, j was available before category v was processed and remains available after it was processed although $j \in \mathcal{E}^v$. Thus, as $C^v_{ⓜ}(\rho;.)$ is maximal for accommodation of HR protections for category v by Lemma A.4 and Eq. (4) holds by definition, we have $\eta^v\big(C^v_{\circledS}(\rhd, \rho; I)\big) = \eta^v\big(C^v_{\circledS}(\rhd, \rho; I) \cup \{j\}\big)$. Thus, the SMH choice rule maximally accommodates HR protections.

No justified envy: Suppose $v \in \mathcal{V}$ and $i \in C^v_{\circledS}(\rhd, \rho; I)$ and $j \in \big(I \cap \mathcal{E}^v\big) \setminus \widehat{C}_{\circledS}(\rhd, \rho; I)$ such that $\sigma_j > \sigma_i$. By Lemma A.4, as $C^v_{ⓜ}(\rho;.)$ satisfies no justified envy we have

$$\eta^v\big(C^v_{ⓜ}(\rho; J)\big) > \eta^v\Big(\big(C^v_{ⓜ}(\rho; J) \setminus \{i\}\big) \cup \{j\}\Big)$$

where $J = I \setminus \bigcup_{\nu \in \mathcal{V}: \nu \lhd v} C^\nu_{\circledS}(\rhd, \rho; I)$. Since $C^v_{\circledS}(\rhd, \rho; I) = C^v_{ⓜ}\big(\rho; J\big)$ (see Eq. (4)), the SMH choice rule satisfies no justified envy. ■

Proof of Lemma 6. Fix $\rho \in \big(2^{\mathcal{R}}\big)^{|\mathcal{I}|}$ and $\rhd \in \Delta^o$. Suppose $I \subseteq \mathcal{I}$, $c \in \mathcal{R}$, and $i \in C^c_{\circledS}(\rhd, \rho; I)$. In executing the procedure of the SMH choice rule according to $\rhd$, the open category is processed first and individual i is still available when the VR-protected category c is about to be processed. Thus, it should be the case that $|C^o_{\circledS}(\rhd, \rho; I)| = q^o$, as otherwise i would have received a category-o position instead of a category-c position. Moreover, $C^o_{ⓜ}(\rho;.)$ satisfies no justified envy by Lemma A.4. Since $C^o_{\circledS}(\rhd, \rho; I) = C^o_{ⓜ}(\rho; I)$, for each $j \in C^o_{\circledS}(\rhd, \rho; I)$, we have $\sigma_j > \sigma_i$ or $\eta^o\big(C^o_{\circledS}(\rhd, \rho; I)\big) > \eta^o\Big(\big(C^o_{\circledS}(\rhd, \rho; I) \setminus \{j\}\big) \cup \{i\}\Big)$. Moreover, $C^o_{ⓜ}(\rho;.)$ maximally accommodates HR protections for the open category also by Lemma A.4, and $i \notin C^o_{ⓜ}(\rho; I) = C^o_{\circledS}(\rhd, \rho; I)$. Thus,

$$\eta^o\big(C^o_{\circledS}(\rhd, \rho; I) \cup \{i\}\big) \not> \eta^o\big(C^o_{\circledS}(\rhd, \rho; I)\big).$$

These show that the SMH choice rule satisfies compliance with VR protections. ■

B.3. Proofs of Results in Section 4.3.2.

Consider the category of memberships $\mathring{\rho} = (\mathring{\rho}_i)_{i \in \mathcal{I}} \in \big(2^{\mathcal{R}}\big)^{|\mathcal{I}|}$ with $|\mathring{\rho}_i| \leq 1$ for each $i \in \mathcal{I}$ so that the VR-protected categories are non-overlapping. Recall that $\mathcal{J} \subseteq \bigcup_{c \in \mathcal{R}^0} \mathcal{E}^c(\mathring{\rho})$ and $\tilde{\rho}_i = \{e\}$ for each $i \in \mathcal{J}$.

We state and prove a more detailed version of Lemma 8 as we use this new lemma also in the proof of Theorem 4.

Lemma B.5. *Fix $I \subseteq \mathcal{I}$. Define $\overline{J}$ as*

$$\overline{J} = (\mathcal{J} \cap I) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I). \tag{5}$$

and J as

$$J = \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-\overline{J}}, \tilde{\rho}_{\overline{J}}); I\big) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I). \tag{6}$$

Then, J is the unique maximal set of individuals who suffer from a violation of the Equality Code under the choice rule $C^{2s}_{Ⓜ}(\mathring{\rho}; .)$ for I.

Proof of Lemma B.5. First observe that Lemma 7 implies for any $K \subseteq \mathcal{J}$,

$$C^{2s,o}_{Ⓜ}\big((\mathring{\rho}_{-K}, \tilde{\rho}_K); I\big) = C^{2s,o}_{Ⓜ}(\mathring{\rho}; I). \tag{7}$$

Then, by the definition of the procedure of the 2SMH choice rule and non-overlapping nature of VR-protected categories at $\mathring{\rho}$ and $(\mathring{\rho}_{-K}, \tilde{\rho}_K)$ that only differ in the memberships of individuals in K for any fixed $K \subseteq (\mathcal{J} \cap I) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I)$, we have

$$C^{2s,c}_{Ⓜ}\big((\mathring{\rho}_{-K}; \tilde{\rho}_K), I\big) = C^{2s,c}_{Ⓜ}(\mathring{\rho}; I) \quad \text{for each } c \in \mathcal{R}^0. \tag{8}$$

As (i) single-category meritorious horizontal rule used in the definition of the 2SMH choice rule satisfies irrelevance of rejected individuals by Lemma A.3 and (ii) category e is processed after the open category

$$\begin{aligned} C^{2s,e}_{Ⓜ}\big((\mathring{\rho}_{-K}, \tilde{\rho}_K); I\big) &= C^{e}_{ⓜ}\big((\mathring{\rho}_{-K}, \tilde{\rho}_K); I \setminus C^{2s,o}_{Ⓜ}(\mathring{\rho}; I)\big) \\ &= C^{e}_{ⓜ}\big((\mathring{\rho}_{-K}, \tilde{\rho}_K); K \cup \widetilde{J}\big) \end{aligned} \tag{9}$$

where we define

$$\widetilde{J} = \mathcal{E}^e(\mathring{\rho}) \cap \big(I \setminus C^{2s,o}_{Ⓜ}(\mathring{\rho}; I)\big).$$

Here, $\widetilde{J}$ is the set of original members of the VR-protected category e in I who do not receive an open category position (i.e., not matched under 2SMH choice rule in Step 1 of its procedure). Thus, Eq. (9) holds as members of sets K and $\widetilde{J}$ are the only individuals who are eligible to be chosen under category e in Step 2 of the procedure of the 2SMH choice rule just before e is processed. Eqs. (7), (8), and (9) imply

$$\widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-K}, \tilde{\rho}_K); I\big) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I) = C^{e}_{ⓜ}\big((\mathring{\rho}_{-K}, \tilde{\rho}_K); K \cup \widetilde{J}\big) \setminus C^{e}_{ⓜ}\big((\mathring{\rho}_{-K}, \tilde{\rho}_K); \widetilde{J}\big). \tag{10}$$

Therefore, by definition of J in Eq. (6) and by Eq. (10) by setting $K = \overline{J}$, we obtain J is also equal to

$$J = C^{e}_{ⓜ}\big((\mathring{\rho}_{-\overline{J}}, \tilde{\rho}_{\overline{J}}); \overline{J} \cup \widetilde{J}\big) \setminus C^{e}_{ⓜ}(\mathring{\rho}; \widetilde{J}). \tag{11}$$

Since the single-category meritorious horizontal choice rule satisfies irrelevance of rejected individuals by Lemma A.3 and $\overline{J} \setminus J$ is a subset of individuals who do not receive any position at membership profile $(\mathring{\rho}_{-\overline{J}}, \tilde{\rho}_{\overline{J}})$ (by definitions in Eqs. (5) and (6)), making these individuals ineligible for category e will not change the choice for this category, i.e., $C^e_{ⓜ}\big((\mathring{\rho}_{-J}, \tilde{\rho}_J); J \cup \widetilde{J}\big) = C^e_{ⓜ}\big((\mathring{\rho}_{-\overline{J}}, \tilde{\rho}_{\overline{J}}); \overline{J} \cup \widetilde{J}\big)$. Hence, Eq. (11) implies that

$$\begin{aligned} J &= C^e_{ⓜ}\big((\mathring{\rho}_{-J}, \tilde{\rho}_J); J \cup \widetilde{J}\big) \setminus C^e_{ⓜ}(\mathring{\rho}; \widetilde{J}) \\ &= \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-J}, \tilde{\rho}_J); I\big) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I), \end{aligned}$$

where the last equality follows from Eq. (10). Thus, J is a set of individuals that suffer from a violation of the Equality Code. Moreover, irrelevance of rejected individuals also implies that for any $J' \subseteq (\mathcal{J} \cap I) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I)$ such that $J \subsetneq J'$, by Eqs. (10) and (11) we similarly have

$$J = \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-J'}, \tilde{\rho}_{J'}); I\big) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I) \subsetneq J';$$

thus, there exists some $i \in J'$ such that $i \notin \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-J'}, \tilde{\rho}_{J'}); I\big)$, so J' is not a set of individuals that suffer from a violation of the Equality Code, and moreover, $J \subseteq \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-J'}, \tilde{\rho}_{J'}); I\big)$. These two establish that J is a maximal set of individuals who suffer from a violation of the Equality Code.

Finally, we prove its uniqueness. Since $\overline{J}$ is a superset of any $J'' \neq J$ such that $J'' \subseteq \mathcal{J} \cap I$ is a set of individuals who suffer from a violation of the Equality Code, Eq. (6) implies that J'' cannot be maximal, as there exists some $i \in J''$ such that $i \notin \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-\overline{J}}, \tilde{\rho}_{\overline{J}}); I\big)$. Therefore, this establishes that J is the unique maximal set of individuals who suffer from a violation of the Equality Code under $C^{2s}_{Ⓜ}(\mathring{\rho}; .)$ for I. ■

Proof of Lemma 8. It directly follows from Lemma B.5. ■

Recall that $\rho^*_i = \mathring{\rho}_i \cup \{e\}$ for each $i \in \mathcal{J}$ and $\rho^*_i = \mathring{\rho}_i$ for each $i \in \mathcal{I} \setminus \mathcal{J}$. Consider the order of precedence $\trianglerighteq$ that orders category o first and category e last.

Proof of Proposition 3. Fix a set of individuals $I \subseteq \mathcal{I}$. Lemma 5 implies that $C_{ⓢ}(\trianglerighteq, \rho^*; .)$ satisfies non-wastefulness, no justified envy, and maximal accommodation of HR protections. Lemma 6 implies that $C_{ⓢ}(\trianglerighteq, \rho^*; .)$ satisfies compliance with VR protections.

Fix $J \subseteq \mathcal{J} \cap I$. We show that J is not a set of individuals that suffer from a violation of the Equality Code under $C_{ⓢ}(\trianglerighteq, \rho^*; .)$. If there exists some $i \in J \cap \widehat{C}_{ⓢ}(\trianglerighteq, \rho^*; I)$ then we are done. So assume that $J \cap \widehat{C}_{ⓢ}(\trianglerighteq, \rho^*; I) = \emptyset$. We use induction in our proof to show that $\widehat{C}_{ⓢ}\big(\trianglerighteq, (\rho^*_{-J}, \tilde{\rho}_J); I\big) = \widehat{C}_{ⓢ}(\trianglerighteq, \rho^*; I)$.

Suppose that as the inductive assumption, for any $J' \subsetneq J$ with $|J'| \leq k$ for a fixed k with $|J| > k \geq 0$ we have $\widehat{C}_{Ⓢ}\big(\trianglerighteq, (\rho^*_{-J'}, \tilde{\rho}_{J'}); I\big) = \widehat{C}_{Ⓢ}(\trianglerighteq, \rho^*; I)$. (For $k = 0$, we have $J' = \emptyset$ in the initial step, and the inductive assumption is vacuously proven for this step.) We prove this statement for $k + 1$.

Let $J' \subsetneq J$ be such that $|J'| = k$ and let $i \in J \setminus J'$. We prove the statement holds for $J' \cup \{i\}$. When it is turn of category e to be processed in Step 2 of the procedure of the SMH choice rule at both $(\rho^*_{-J'\cup\{i\}}, \tilde{\rho}_{J'\cup\{i\}})$ and $(\rho^*_{-J'}, \tilde{\rho}_{J'})$, as i is not selected yet, the same set of individuals are selected until that point at both cases, as only i's category membership is different at both profiles. Moreover, as i is not selected at $(\rho^*_{-J'}, \tilde{\rho}_{J'})$ by the inductive assumption, she will not receive a position at $(\rho^*_{-J'\cup\{i\}}, \tilde{\rho}_{J'\cup\{i\}})$, either, as $\tilde{\rho}_i \subseteq \rho^*_i$. Thus, $i \notin \widehat{C}_{Ⓢ}(\trianglerighteq, (\rho^*_{-J'\cup\{i\}}, \tilde{\rho}_{J'\cup\{i\}}); I)$, and moreover, $\widehat{C}_{Ⓢ}\big(\trianglerighteq, (\rho^*_{-J'}, \tilde{\rho}_{J'}); I\big) = \widehat{C}_{Ⓢ}\big(\trianglerighteq, (\rho^*_{-J'\cup\{i\}}, \tilde{\rho}_{J'\cup\{i\}}); I\big)$ completing the inductive step's proof.

We showed that $\widehat{C}_{Ⓢ}\big(\trianglerighteq, (\rho^*_{J}, \tilde{\rho}_{J}); I\big) = \widehat{C}_{Ⓢ}(\trianglerighteq, \rho^*; I)$ proving that $J \not\subseteq \widehat{C}_{Ⓢ}\big(\trianglerighteq, (\rho^*_{J}, \tilde{\rho}_{J}); I\big)$, and hence, J is not a set of individuals who suffer from a violation of the Equality Code. This proves that $C_{Ⓢ}(\trianglerighteq, \rho^*; .)$ abides by the Equality Code for I. ■

Proof of Theorem 4. Fix a set of individuals $I \subseteq \mathcal{I}$. Define

$$J = \widehat{C}_{Ⓢ}(\trianglerighteq, \rho^*; I) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I).$$

We prove below in Claim B.1

$$\widehat{C}_{Ⓢ}(\trianglerighteq, \rho^*; I) = \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-\bar{J}}, \tilde{\rho}_{\bar{J}}); I\big),$$

where

$$\bar{J} = \big(\mathcal{J} \cap I\big) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I),$$

so that the rest of the proof follows from Lemma B.5.

Claim B.1. $\widehat{C}_{Ⓢ}(\trianglerighteq, \rho^*; I) = \widehat{C}^{2s}_{Ⓜ}\big((\mathring{\rho}_{-\bar{J}}, \tilde{\rho}_{\bar{J}}); I\big)$.

Proof of Claim B.1. Let

$$\rho' = (\mathring{\rho}_{-\bar{J}}, \tilde{\rho}_{\bar{J}}).$$

Observe that for individuals in I, we have

(1) for each $i \in \mathcal{J} \cap \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I)$, $\quad \rho^*_i = \mathring{\rho}_i \cup \{e\} \quad \& \quad \rho'_i = \mathring{\rho}_i$,
(2) for each $i \in \big(\mathcal{J} \cap I\big) \setminus \widehat{C}^{2s}_{Ⓜ}(\mathring{\rho}; I)$, $\quad \rho^*_i = \mathring{\rho}_i \cup \{e\} \quad \& \quad \rho'_i = \{e\}$,
(3) for each $i \in I \setminus \mathcal{J}$, $\quad \rho^*_i = \mathring{\rho}_i \quad \& \quad \rho'_i = \mathring{\rho}_i$.

Lemma 7 implies that

$$C^{o}_{Ⓢ}(\trianglerighteq, \rho^*; I) = C^{2s,o}_{Ⓜ}(\rho'; I) = C^{2s,o}_{Ⓜ}(\mathring{\rho}; I). \tag{12}$$

Since VR-protected categories other than e do not overlap with each other at ρ^* and VR-protected categories do not overlap at all at ρ' and $\mathring{\rho}$, their order of precedence does not matter for $C^{2s}_{Ⓜ}$ under ρ' and $\mathring{\rho}$ and as long as e is processed last as it is done for choice rule $C_{Ⓢ}(\trianglerighteq, \rho^*; .)$ under $\trianglerighteq$. Therefore, for each $c \in \mathcal{R}^0$,

$$C^c_{Ⓢ}(\trianglerighteq, \rho^*; I) = C^{2s,c}_{Ⓜ}(\rho'; I) = C^{2s,c}_{Ⓜ}(\mathring{\rho}; I). \tag{13}$$

Thus, only the set of individuals who receive category-e positions could possibly differ under both choice rules $C^{2s,e}_{Ⓜ}(\rho'; .)$ and $C^e_{Ⓢ}(\trianglerighteq, \rho^*; .)$. By Eqs. (12) and (13), in the procedures of $C^{2s}_{Ⓜ}(\rho'; .)$ and $C_{Ⓢ}(\trianglerighteq, \rho^*; .)$ just before category e is processed, we have exactly the same set of eligible individuals available for category e by definitions of $\overline{J}$ and ρ'. Since each choice rule uses the single-category meritorious choice rule for category e

$$C^e_{Ⓢ}(\trianglerighteq, \rho^*; I) = C^{2s,e}_{Ⓜ}(\rho'; I). \tag{14}$$

Eqs. (12), (13), and (14) imply

$$\widehat{C}_{Ⓢ}(\trianglerighteq, \rho^*; I) = \widehat{C}^{2s}_{Ⓜ}(\rho'; I).$$

□

■

B.4. Proof of Result in Section 4.3.3.

Proof of Theorem 5. Fix $\rho = (\rho_i)_{i \in \mathcal{I}} \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$ such that for any $i \in \mathcal{I}$, $|\rho(i)| \leq 2$ and $|\rho(i)| = 2$ implies $e \in \rho(i)$. Fix also $\overline{\triangleright} \in \Delta^{o,e}$. Let choice rule $C(\rho; .)$ maintain the elevated status of caste-based VR protections and satisfy non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections.

Let $I \subseteq \mathcal{I}$. By Lemma 7, for any profile of non-overlapping category memberships $\rho' \in \left(2^{\mathcal{R}}\right)^{|\mathcal{I}|}$, $C^o(\rho, I) = C^{2s,o}_{Ⓜ}(\rho', I)$ and $C^o_{Ⓢ}(\overline{\triangleright}, \rho; I) = C^{2s,o}_{Ⓜ}(\rho', I)$. Thus,

$$C^o(\rho, I) = C^o_{Ⓢ}(\overline{\triangleright}, \rho; I) \tag{15}$$

Let

$$J = I \setminus C^o(\rho, I).$$

We prove the following claim:

Claim B.2. $C^e(\rho, I) = C^e_{Ⓢ}(\overline{\triangleright}, \rho; I)$.

Proof of Claim B.2. Recall that $C^e_{Ⓢ}(\overline{\triangleright}, \rho; I) = C^e_{ⓜ}(\rho; J)$. To show that $C^e(\rho; I) = C^e_{ⓜ}(\rho; J)$, we consider component $C^e(\rho; I)$ as a single-category choice rule executed on set J to invoke Lemma A.4. To this end, $C^e(\rho; I)$ satisfies the following three properties:

$C^e(\rho; I)$ satisfies non-wastefulness on J: Let $j \in J \setminus C^e(\rho; I)$. Suppose $|C^e(\rho; I)| < q^e$. We show that $j \notin \mathcal{E}^e(\rho)$ to complete the proof.

Since $C(\rho; .)$ is non-wasteful, either (i) $j \notin \mathcal{E}^e(\rho)$ or (ii) $j \in \mathcal{E}^e(\rho)$ and $j \in \widehat{C}(\rho; I)$.

We show that (ii) does not hold. Contrary to the claim, suppose it does. As $j \notin C^o(\rho; I)$ by $j \in J$ and Eq. (15), then $j \in C^c(\rho, I)$ for some $c \in \mathcal{R}^0$. Since $C(\rho; .)$ maintains the elevated status of caste-based VR protections then $|C^e(\rho; I)| = q^e$, a contradiction. Therefore, we showed that (ii) cannot hold. Thus, $j \notin \mathcal{E}^e(\rho)$.

$C^e(\rho; I)$ satisfies no-justified envy on J: Let $i \in C^e(\rho; I)$ and $j \in (\mathcal{E}^e(\rho) \cap J) \setminus C^e(\rho; I)$. Suppose

$$\eta^e\big(C^e(\rho; I)\big) \leq \eta^e\Big(\big(C^e(\rho; I) \setminus \{i\}\big) \cup \{j\}\Big). \tag{16}$$

We show that $\sigma_i > \sigma_j$ to complete the proof.

If $j \notin \widehat{C}(\rho; I)$ then no-justified envy property of $C(\rho; .)$ implies $\sigma_i > \sigma_j$. Consider the other case, $j \in \widehat{C}(\rho; I)$. Eq. (15) implies that $j \in C^c(\rho; I)$ for some $c \in \mathcal{R}^0$. Since $C(\rho; .)$ maintains the elevated status of caste-based VR protections and $i \in C^e(\rho; I)$, we have either (i) $\sigma_i > \sigma_j$ or (ii) $\eta^e\big(C^e(\rho; I)\big) > \eta^e\Big(\big(C^e(\rho; I) \setminus \{i\}\big) \cup \{j\}\Big)$. By Eq. (16), Case (ii) cannot hold. Therefore, $\sigma_i > \sigma_j$.

$C^e(\rho; I)$ satisfies maximal accommodation with HR protections on J: Let $i \in (\mathcal{E}^e(\rho) \cap J) \setminus C^e(\rho; I)$. We show that

$$\eta^e\big(C^e(\rho; I) \cup \{i\}\big) \not> \eta^e\big(C^e(\rho; I)\big) \tag{17}$$

to complete the proof.

If $i \notin \widehat{C}(\rho; I)$, then by the fact that $C(\rho; .)$ satisfies maximal accommodation with HR protections, Eq. (17) holds. If $i \in \widehat{C}(\rho; I)$, then by Eq. (**??**) implies that $i \in C^c(\rho; I)$ for some $c \in \mathcal{R}^0$. Since $C(\rho; .)$ maintains the elevated status of caste-based VR protections, Eq. (17) holds.

Lemma 0 implies $C^e(\rho; I) = C^e_{ⓜ}(\rho; J) = C^e_{Ⓢ}(\rhd, \rho; I)$ □

Let

$$K = J \setminus C^e(\rho; I).$$

Let $c \in \mathcal{R}^0$. Since $|\rho_i| \leq 2$ and $|\rho_i| = 2$ implies $e \in \rho_i$ for each $i \in \mathcal{I}$, and both categories o and e precede all categories in $\mathcal{R}^0$, precedence order of categories in $\mathcal{R}^0$ is immaterial to the outcome of $C_{Ⓢ}$. In particular, $C^c_{Ⓢ}(\rhd, \rho; I) = C^c_{ⓜ}(\rho; K)$. To show that $C^c(\rho; I) = C^c_{ⓜ}(\rho; K)$, we consider execution of the component $C^c(\rho; I)$ as a single-category choice rule on K. Given Eq. (15) and Claim B.2, non-wastefulness, no justified envy, and maximal accommodation of HR reservations properties of $C(\rho; .)$ directly imply the corresponding properties are satisfied by $C^c(\rho; I)$ on K. Lemma 0 implies that $C^c(\rho; I) = C^c_{ⓜ}(\rho; K) = C^c_{Ⓢ}(\overline{\triangleright}, \rho; I)$. This together with Eq. (15) and Claim B.2 imply that $C(\rho; I) = C_{Ⓢ}(\overline{\triangleright}, \rho; I)$. ■