ANALYSIS QUALIFYING EXAM

SEPTEMBER , 2012

REAL ANALYSIS

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1 (30 points)
Let (X, M, μ) be a measure space. A measure μ is **semi-finite** if for each $E \in M$, with $\mu(E) = \infty$, there exists an $F \in M$ such that $0 < \mu(F) < \infty$.
Prove that if μ is semifinite and $\mu(E) = \infty$, for any $C > 0$ there exists an $F \in M$ such that $C < \mu(F) < \infty$.

Question 2 (20 points)
Let (X, M, μ) be a measure space and $L^p(X) = \{f : X \to \mathbb{C} \ : \ f \text{ is measurable and } ||f||_p < \infty\}$.
Prove that $L^p(X)$ is a Banach space for $1 \leq p < \infty$ by proving
a) If $f, g \in L^p(X)$ then $||f + g||_p \leq ||f||_p + ||g||_p$
b) $L^p(X)$ is complete.

Question 3 (30 points)
The **total variation** of a complex measure ν is the positive measure $|\mu|$ determined by the property that if $d\nu = f d\mu$ for some positive measure μ, $f \in L^1(\mu)$, then $d|\nu| = |f| d\mu$.
Prove that this is well defined by showing the following;
a) There always exists such a measure μ.
b) The definition is independent of μ.

Question 4 (20 points)
a) Let $||.||_1, ||.||_2$ be two norms on a vector space V such that $||v||_1 \leq ||v||_2$ for all $v \in V$. If V is complete with respect to both norms, prove that they are equivalent.
b) Let X, Y be Banach spaces and let $T_n \in L(X, Y)$ such that $T(x) = \lim_{n\to\infty} T_n(x)$ exists for all $x \in X$. Prove that $T \in L(X, Y)$.

COMPLEX ANALYSIS

You should attempt all the problems. Partial credit will be give for serious efforts

(1) Compute the following integral:

$$\int_0^\infty \frac{\log x}{x^2+1}\, dx$$

(2) Let $\mathbb{A} = \{a_0, a_1, \ldots a_n\}$ be a finite set of (distinct) points in the unit disk D. Define

$$B_{\mathbb{A}}(z) = \Pi_{i=0}^{n} \frac{z - a_i}{1 - \overline{a}_i z} \frac{|a_i|}{a_i}, \qquad \text{for } z \in D$$

where if $a_i = 0$, we set $\frac{|a_i|}{a_i} = 1$.

(a) Prove that $B(z)$ maps D to D and maps the unit circle to the unit circle.

(b) Let $T : D \to D$ be a fractional linear transformation that maps the unit disk onto itself. Prove that

$$B_{\mathbb{A}} \circ T = \lambda B_{T^{-1}(\mathbb{A})}$$

where λ is a constant with $|\lambda| = 1$ and $T^{-1}(\mathbb{A}) = \{T^{-1}(a_0), \ldots, T^{-1}(a_n)\}$.

(c) Let $f : D \to D$ be an analytic function with $f(a_i) = 0$ for each $a_i \in \mathbb{A}$. Prove that $|f(z)| \leq |B_{\mathbb{A}}(z)|$ for all $z \in D$.

(3) The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the *Schwarzian derivative of f*. If $f(z)$ has a zero or pole of order m $(m > 1)$ at z_0, show that $\{f, z\}$ has a pole at z_0 of order 2 and calculate the coefficient of $\frac{1}{(z-z_0)^2}$ in the Laurent development of $\{f, z\}$.

(4) Let f be a bounded analytic function on the unit disk $|z| < 1$ and let ζ be a point in the unit disk (i.e. $|\zeta| < 1$)

(a) Show that the area integral

$$\iint_{|z|<1} \frac{f(z)\,\mathrm{d}x\,\mathrm{d}y}{(1-\bar{z}\zeta)^2}, \qquad z = x + yi$$

is equal to

$$\int_0^1 \left(\int_{|z|=1} \frac{z f(rz)}{i(z-r\zeta)^2}\,\mathrm{d}z \right) r\,\mathrm{d}r$$

(Hint: use polar coordinates)

(b) Use part (a) to prove

$$f(\zeta) = \frac{1}{\pi} \iint_{|z|<1} \frac{f(z)\,\mathrm{d}x\,\mathrm{d}y}{(1-\bar{z}\zeta)^2}$$