Algebra Qualifying Exam
August, 2012

Please answer all 10 problems and show your work. Each problem is worth 20 points. In your proofs, you may use any theorem from the syllabus for Algebra, except of course you may not use the fact you are trying to prove, or a mere variant of it. State clearly what theorems you use. Good luck.

1. Let G be a group of order pqr, where $p < q < r$ are distinct primes. Prove that a Sylow r-subgroup of G is normal in G.

2. Describe the irreducible representations of the dihedral group of order 12, and give their dimensions.

3. Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} -1 & -5 & 2 \\ 0 & 1 & 0 \\ -2 & -5 & 3 \end{bmatrix}.$$

4. Find representative matrices for the conjugacy-classes of elements of order six in $\mathrm{GL}_3(\mathbb{Q})$.

5. Let F be a finite field of cardinality q and let V be a four-dimensional vector space over F. The group $\mathrm{GL}(V) \simeq \mathrm{GL}_4(F)$ acts on V. Let U be a two-dimensional subspace of V. Compute the order of the subgroup $\{g \in GL(V) : \ gU = U\}$ and determine the number of two-dimensional subspaces of V.

6. Let R be a commutative ring with identity and I, J two ideals in R. Prove that

$$\mathrm{Tor}_1^R(R/I, R/J) \simeq (I \cap J)/IJ.$$

7. Let $R = F[x]$ and $S = K[x]$ be polynomial rings over fields F and K, where F is a subfield of K. Suppose M and N are finitely generated R-modules such that $S \otimes_R M \simeq S \otimes_R N$, as S-modules. Prove that $M \simeq N$ as R-modules.
[Hint: Every proper ideal in R or S is generated by a monic polynomial.]

8. Consider the following situation: E/F is a Galois degree 3 extension of fields and K/F is a NON-Galois degree 4 extension of fields and the compositum KE is Galois over F. Either prove that such a situtation is impossible, or give an example of such F, E, K and prove that your example works.

9. Let R be a noetherian commutative ring with identity and E a finitely generated R-module. Let M be one of the maximal ideals of R and suppose that F is a submodule of E such that $ME_M + F_M = E_M$. Prove that there exists $x \notin M$ such that $xE \subset F$.

10. Let S be an integral domain, G a finite group acting on S by ring automorphisms and $R = S^G$, the ring of invariants.

(a) Prove that S is integral over R.

(b) Let k be a field and suppose that S is a finitely generated k-algebra, and that G acts on S via k-algebra automorphisms. Prove that S is a finitely generated R-module.