ALGEBRAIC TOPOLOGY QUALIFYING EXAM

Write your answers on the test pages. Show all your work and explain all your reasoning. You may use any standard result, as long as you state clearly what result you are using (including its hypotheses). Exception: you may not use a result which is the same as the problem you are being asked to do. Each problem has a noted value, in total 40 *points.*

Name: ______________________________

Date: August 25, 2017.

1. (10 points) Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the integer homology groups of $S(X)$ in terms of the homology groups of X.

2. (10 points) Let X be the space obtained from identifying three distinct points on the 2-sphere S^2. Compute the integer homolog groups of X.

3. (10 points) Show that if $f : \mathbb{RP}^{2n} \to X$ is a covering map of a CW-complex X, then f is a homeomorphism.

4. (10 points) Let M be a closed, connected, orientable real n-dimensional manifold and $f : S^n \to M$ a continuous map such that the induced morphism $f_* : H_n(S^n;\mathbb{Z}) \to H_n(M;\mathbb{Z})$ is non-trivial. Calculate $H_k(M;\mathbb{Q})$ for all k.

Differential Topology Qual

June 8, 2017

Write your answers on the test pages. Show all work and explain all reasoning. You may use results from the class or course notes. Please write legibly!! Oh, and don't forget to put your name on the exam.

You must do Problems 1,2,3. You must do *either* Problem 4 or Problem 5. Please indicate clearly which of these last two you would like to be graded.

Problem 1. Prove that n-sphere

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is a smooth manifold.

Problem 2. Suppose M is a smooth manifold.

- What is a tangent vector to M? What is a cotangent vector?
- What are the tangent and cotangent bundles TM and T^*M?
- What is a Riemannian metric on M?
- Explain how a Riemannian metric can be used to define a natural isomorphism between TM and T^*M.

Problem 3. Suppose G is an n-dimensional Lie group. Recall that a vector field v on G is left-invariant if $(L_g)_*(v_x) = v_{gx}$ for all $g, x \in G$, where $(L_g)_*$ denotes the differential of the map

$$L_g : G \to G$$

given by left-multiplication by g.

- Show that the set of left-invariant vector fields forms an n-dimensional vector space.
- Prove that the tangent bundle TG is trivial.

Problem 4. Prove that there is a smooth vector field on S^2 which vanishes at exactly one point. (Hint: consider using the stereographic projection.)

Problem 5. Let ω be the $(n-1)$-form on $\mathbb{R}^n \smallsetminus \{0\}$ given by

$$\omega = ||x||^{-n} \cdot \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Show that ω is closed but not exact on $\mathbb{R}^n \smallsetminus \{0\}$. (Hint: for the latter, what happens if you integrate this form over $S^{n-1} \subset \mathbb{R}^n$?)