REAL ANALYSIS QUALIFYING EXAM

AUG 2017

Answer all 4 questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Exercise 1. (25 points). Suppose $(X, \mathcal{M}, \mu)$ is a measure space.

(1) What does it mean for a function $f : X \to \mathbb{R}$ to be measurable?
(2) State the monotone convergence theorem for this measure space
(3) State and prove Fatou's Lemma, relating $\int \liminf f_n d\mu$ and $\liminf \int f_n d\mu$ [Hint: Consider the functions $g_n(x) = \inf_{j \geq n} f_j(x)$, and use the monotone convergence theorem appropriately]

Exercise 2. (25 points).
Let μ, ν be finite measures on a measure space $(X, \mathcal{M})$ and let $\lambda = \mu + \nu$.

(1) Define the notion of an absolutely continuous measure and show that ν is absolutely continuous with respect to λ.
(2) State the Radon-Nikodym theorem with respect to ν and λ and define the Radon-Nikodym derivative $f = \frac{d\nu}{d\lambda}$.
(3) Assme that ν is absolutely continuous with respect to μ and show that $0 \leq f < 1$ μ-a.e. and that $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

Exercise 3. (25 points).
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^\infty(X) \cap L^1(X)$.

(1) Define the norms $\|f\|_\infty$ and $\|f\|_p$ and the spaces $L^\infty(X)$ and $L^p(X)$ for $1 \leq p < \infty$.
(2) Show that $f \in L^p(X)$ for all $1 \leq p < \infty$ and that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$.

Exercise 4. (25 points).
Let (f_n) be a sequence of real valued Lebesgue measurable functions on $\mathbb{R}$. For each statement, prove or give a counter example:

(1) If every subsequence f_{n_k} has subsequence $f_{n_{k_l}}$ that converges in L^1 to 0, then the original sequence converges in L^1.
(2) If every subsequence f_{n_k} has subsequence $f_{n_{k_l}}$ that converges almost everywhere to 0, then the original sequence converges almost everywhere.

COMPLEX ANALYSIS QUALIFYING EXAM

Write your answers on the test pages. Show all your work and explain all your reasoning. You may use any standard result, as long as you state clearly what result you are using (including its hypotheses). Exception: you may not use a result which is the same as the problem you are being asked to do. Each problem has a noted value, in total 40 *points.*

Name: ______________________________

Date: August 25, 2017.

1. (10 points) (a) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function such that both f^2 and f^3 are entire. Prove that f is entire.

(b) Suppose f is an analytic function on $\{z : 0 < |z| < 1\}$ and $|f(z)| \leq \log(1/|z|)$. Prove that f is identically zero.

2. (10 points) Evaluate the following integral:

$$\int_0^{\pi} \frac{d\theta}{5 + 3\cos\theta}.$$

3. (10 points) Suppose f and g are entire functions such that $|f(z)| < |g(z)|$ for all z with $|z| \geq R$ for a constant R. Prove that f/g is a rational function.

4. (10 points) Let X be a closed and connected Riemann surface. Let ω be a meromorphic one-form on X. Prove that the sum of residues of ω is zero.