ALGEBRA QUALIFYING EXAM – SPRING 2017

Problem 1. Prove that an Artinian ring has finitely many maximal ideals.

Problem 2. Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = q$. Consider the subgroup

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}^\times,\ b \in \mathbb{F} \right\} < \mathrm{GL}_2(\mathbb{F}).$$

Show that for any prime p dividing $q - 1$, the number of Sylow p-subgroups of G is q.

Problem 3. Let R be a UFD and a, b be coprime elements in R. For all $i \geq 0$, compute

$$\mathrm{Tor}_i^{R/(ab)}(R/(a), R/(b)).$$

Problem 4. Let F be a field, and D be an integral domain containing F. Suppose D is finite dimensional as a vector space over F. For each $x \in D$, define the F-linear transformation $T_x\colon D \to D$ by $T_x(y) = xy$.
(a) Prove that D is a field.
(b) Suppose $p = \mathrm{char}(F) > 0$ and $\alpha \in D$ is purely inseparable over F. This means that the minimal polynomial of α over F is $T^{p^e} - r$ for some $r \in F$ and $e \geq 1$. Describe the Jordan canonical form of T_α over the algebraic closure of F.

Problem 5. Let K be a field of characteristic $p > 0$ and $F = K(t)$ where t is a variable. Let $f(x) = x^{2p} - tx^p + t \in F[x]$.
(a) Show that $f(x)$ is irreducible in $F[x]$.
(b) Let $E = F[s]$ where s is a root of the polynomial $(x^p - t) \in F[x]$. If L is the splitting field of $f(x)$ over E, show that $[L : E] \leq 2$.
(c) Show that $L = F[\alpha]$, where α is a root of $f(x)$.

Problem 6. Prove that a flat finitely-generated module over a Noetherian local ring is free.

Problem 7. Let p be a prime integer, and q be a power of p. Let $\mathbb{F}_q$ be the finite field with q elements, and $\mathbb{F}_{q^n}$ be the degree n extension of $\mathbb{F}_q$. Consider the map $N\colon \mathbb{F}_{q^n} \to \mathbb{F}_q$ defined by $N(x) = x^{1+q+\cdots+q^{n-1}}$.
(a) Prove that N is surjective. (*Hint*: Recall that $\mathbb{F}_{q^n}^*$ is a cyclic group of order $q^n - 1$.)
(b) Prove that $N^{-1}(1)$ spans $\mathbb{F}_{q^n}$ as an $\mathbb{F}_q$-vector space.

Problem 8. Suppose k is a field. Let $R = k[s^4, s^3t, st^3, t^4] \subset k[s,t]$.

(1) Compute the Krull dimension of R.
(2) Prove that R is not Cohen-Macaulay. (*Hint*: Consider R/s^4R.)

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