Algebra Qualifying Examination, June 2019

Instuctions: This is a 3 hour examination. In the problems below, all rings are commutative with identity. This is a closed book exam, also no notes, searching the web, or otherwise consulting external sources. Good luck!

1. Let G be a group of order 108. Show that G has a normal subgroup of order 9 or 27.

2. Let R be a ring, and let $\mathcal{D}$ be the set of all $x \in R$ such that x is a zero divisor or $x = 0$. Show that $\mathcal{D}$ is a union of prime ideals. (Hint: consider the set Σ of all ideals contained in $\mathcal{D}$. Show that Σ contains maximal elements and every maximal element of Σ is prime.)

3. a) Suppose that V is a finite dimensional vector space over a field F and $T \in \mathrm{End}_F(V)$. Show that the characteristic polynomial of T is irreducible over F if and only if V has no nontrivial proper T-invariant subspaces.

   b) Let V be a 3-dimensional vector space over $\mathbb{F}_5$, the field with 5 elements. Give an example of a linear transformation of V that does not have a proper T-invariant subspace.

4. Let p be a prime number and let F be a field of characteristic 0. Suppose that every finite extension of F has degree divisible by p. Show that in fact every finite extension of F has degree a power of p.

5. Let R be a local noetherian ring with maximal ideal M, let A be a finitely generated nonzero R-module, and set $k = R/M$. Prove: $0 < \dim_k(k \otimes_R A) < \infty$.

6. a) Show $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

   b) Is $\mathbb{Q}/\mathbb{Z}$ a projective $\mathbb{Z}$-module? Prove your answer.

7. Let k be a field, $R = k[x, y, z]$ a polynomial ring.

   Set $P_1 = (x, y), P_2 = (x, z), M = (x, y, z), I = P_1P_2$.

   a) Prove that M^2 is a primary ideal in R.

   b) Prove that $I = P_1 \cap P_2 \cap M^2$ is a minimal primary decomposition of I.

8. Let k be a field, and B a finitely generated k-algebra. Suppose B is a field. Prove that $\dim_k(B)$ is finite.