Assortative Matching with Externalities and Farsighted Agents*

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Abstract

We consider a one-to-one assortative matching problem in which matched pairs compete for a prize. With such externalities, the standard solution concept, pairwise stable matching, may not exist. In this paper, we consider farsighted agents and analyze the *largest consistent set* (LCS) of Chwe (1994). Despite the assortative structure of the problem, LCS tend to be large with the standard effectiveness functions: LCS can be the set of all matchings, including an *empty matching with no matched pair*. By modifying the effectiveness function motivated by Knuth (1976), LCS becomes a singleton of the positive assortative matching. Our results suggest that the choice of effectiveness function can significantly impact the solution in a matching problem with externalities.

Keywords: group contest, pairwise stable matching, assortative matching, farsightedness, largest consistent set, effectiveness function

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1 Introduction

There is a large literature on two-sided matching problems after a celebrated paper by Gale and Shapley (1962). The structures and the properties of its central solution concept, pairwise stable matching, have been investigated extensively. At the same time, relatively little attention has been paid to matching problems with externalities, despite their ubiquity in many matching markets in the real world. For instance, matched pairs compete after a matching is formed. In this case, a matched pair cares not only about their partners, but also the other matched pairs.

In this paper, we study a pairs competition problem introduced by Imamura, Konishi, and Pan (2021). A motivating example that we will refer to throughout this paper is a pairs figure skating competition. There are male and female figure skaters who differ in their abilities or skills. They look for partners to participate in a pairs figure skating competition. Once pairs are formed, they then play a Tullock contest based on each pair's aggregated effort with complementarity—each agent makes an effort independently to increase their pair's probability of winning the prize.[1] In this matching problem, agents care not only about their own partners but also other pairs' profiles. In such a matching problem with externalities, it is important for potential deviators to know which matching would be realized after a deviation from a matching. This is specified by an effectiveness function. The standard effectiveness function used in the literature is the one by Roth and Vande Vate (1990): when a pair of agents deviates from a matching, the resulting matching is identical to the original matching except that (1) the deviators are matched, and (2) the deviators' previous partners stay single.[2]

However, if agents are myopic, there may not be a pairwise stable matching in this problem.

[1] Thus, there is a free-rider problem in this pairs competition. For example, if a low-ability male is paired with a high ability female, then a low ability male agent may not make much effort, free-riding on his partner.

[2] An effectiveness function assigns a resulting matching to every combination of matching and a pair deviating from it (Rosenthal 1972). The effectiveness function in Roth and Vande Vate (1990) are commonly used (see Diamantoudi and Xue 2003, Diamantoudi, Miyagawa, and Xue 2004, and Kojima and Unver 2008).

Consider the following example. Suppose that there are three male and three female skaters with high, medium, and low ability. It is natural to predict a positive assortative matching as an outcome of this example. Is it pairwise stable under the above effectiveness function? Consider a deviation by the high ability male and the medium ability female skaters from the assortative matching. Then, according to the effectiveness function, their former partners, the high ability female and the medium ability male, cannot participate in the pairs competition, since they become singles. This means that there are only two pairs in the competition, and the deviating pair gets a high winning probability against the low ability pair. Thus, in the presence of externalities, there may not be a pairwise stable matching under the standard effectiveness function.

When the high ability male and the medium ability female agents deviate, they do not expect any reaction from their former partners. Since single agents cannot participate in the pairs contest, it is beneficial to match with any available partner. Given the two singles dumped by their partners are available to form a pair, it is probably not reasonable for the deviating pair to expect their deviation to decrease the number of pairs. Thus, it is natural to investigate whether or not agents' farsightedness resolves this nonexistence result.

The farsightedness of agents is described by a binary relation "indirect domination" between matchings following Harsanyi (1973) and Chwe (1994). We use the *largest consistent set* (LCS) introduced by Chwe (1994) as a solution concept. LCS has been an in influential solution concept with nice properties such as existence and uniqueness under very mild conditions. In addition, there is a clear result of LCS in a positive assortative matching problem without externalities (Becker 1973): Diamantoudi and Xue (2003) show that LCS is a singleton of the positive assortative matching.[3] LCS tends to be large and includes other solution concepts such as the farsighted

[3]Diamantoudi and Xue (2003) consider LCS in a class of characteristic function games (thus with no externalities across coalitions): a hedonic game is an NTU game in which there is a single payoff vector for each coalition (Banerjee, Konishi, and Sönmez, 2001; Bogomolnaia and Jackson, 2002), and investigated farsighted solutions. Diamantoudi and Xue (2003) show that if a hedonic game satisfies a top-coalition property introduced by Banerjee,

stable sets and farsighted core. Thus, this result tells the assortative matching is a quite robust irrespective of solution concepts in the model without externalities. We investigate how LCS changes in the presence of externalities.

We obtain two main results. First, with the standard effectiveness function by Roth and Vande Vate (1990), we find an example of a pairs competition in which LCS is the set of all matchings, including the *empty matching*, in which all agents are singles. This result stands in stark contrast to the one by Diamantoudi and Xue (2003): once externalities across pairs are allowed, LCS can expand from a singleton to the set of all matchings. Second, we show that LCS becomes a singleton of the positive assortative matching under an alternative effectiveness function proposed by Knuth (1976).[4] According to this effectiveness function, when a pair of agents deviates from a matching, the resulting matching is identical to the original matching except that (1) deviators are matched, and (2) the deviators' former partners are matched, not single. Thus, with the Knuth effectiveness function, we obtain the parallel result of Diamantoudi and Xue (2003). Taken together, our results suggest that the choice of effectiveness function can significantly impact LCS in a matching problem with externalities. In contrast, it is irrelevant in a matching problem without externalities.

The rest of the paper is organized as follows. The rest of this section presents a brief literature review. Section 2 presents the model, and Section 3 presents the results. Section 4 concludes.

Konishi, and Sönmez (2001), which includes Becker's positive assortative matching problem (1973) as a special case, then LCS is equivalent to a singleton set of the core, which is the positive assortative matching in a two-sided one-to-one matching context.

[4] In the companion paper, Imamura, Konishi, and Pan (2021) show that if we use the Knuth effectiveness function via swapping, then there is a unique (myopically) pairwise stable matching (via swapping) which is the assortative matching.

1.1 Literature Review

There are three branches of literature are related to our paper. The first branch is the matching problem with externalities.[5] Recently, a number of papers have been written in this field. Sasaki and Toda (1996) was the first to analyze a one-to-one matching problem with externalities. They considered a set of admissible matchings which can be realized after a pair is formed (or deviates), and defined pairwise stable matching, assuming that deviating pairs expect the worst case scenario. They showed that the admissible set needs to be the set of all matchings to ensure the existence of a stable matching, and proved that there always exists a Pareto-efficient stable matching. Hafalir (2007) imposed certain rationality constraints on players' expecting which set of matchings might be realized by forming a pair, and showed the existence of stable matching under pessimism as in Sasaki and Toda (1996). Chen (2019) considered a specific example of Cournot oligopoly game played by joint ventures, assuming that each pair has unique expectation on the realization of a matching if it is formed. With this list of expectations for each possible pair, each player chooses his/her partner and Chen defined a stable matching as the outcome of this game. Chen identified conditions under which positively and negatively assortative matchings are stable. Mumcu and Saglam (2010) introduced outside options, and Fisher and Hafalir (2016) and Chade and Eeckhout (2020) avoided the impacts of pairwise deviations through externalities by imposing a behavioral assumption and by considering a continuum of atomless agents, respectively. Bando (2012, 2014) and Pycia and Yenmez (2021) considered one-to-many and many-to-many matchings, and analyzed the standard stability concept and its existence by imposing assumptions on agents' preferences.

Second is the field of farsighted stability. Mauleon, Vannetelbosch, and Vergote (2011), Herings, Mauleon, and Vannetelbosch (2020), and Kimya (2021) considered farsighted agents in one-to-one matching problems without externalities. The first two papers showed that every farsighted

[5] More generally, there is a large literature of theory of coalition formation with externalities, starting from Hart and Kurz (1983). For surveys from various aspects, see Bloch (1997), Ray (2008), and Ray and Vohra (2014).

stable set is a singleton set of a stable matching under coalitional and pairwise effectiveness functions, respectively. Kimya (2021) showed that the largest maximal farsighted set in the spirit of Dutta and Vartiainen (2020) coincides with LCS by Chwe (1994) in this domain with coalitional deviations.[6] We consider farsighted agents in the pairs competition model with externalities in this paper, and show that the choice of effectiveness function matters, providing an example where LCS, under the standard effectiveness function, is the set of all matchings, including a fully unmatched matching.

Third, our paper belongs to the literature of assortative matching. Becker (1973) introduced the assortative model of marriages. Banerjee, Konishi, and Sönmez (2001) extended Becker's assortative matching problem to hedonic coalition formation problems without externalities by defining a *top coalition property*. This property guarantees the existence and uniqueness of the core.[7] Diamantoudi and Xue (2003) proved that under the top coalition property, LCS coincides with a singleton core under the standard effectiveness function in coalition formation problems. Mauleon, Vannetelbosch, and Vergote (2011) derived the same result in the context of one-to-one matching. Although our model has the same assortative structure, the results are quite different with externalities.

2 The Model

We first define our one-to-one matching problem with externalities, and introduce basic terminologies in the next subsection, then we move on to introduce (figure skating) pairs competition problem.

[6] Dutta and Vartiainen (2020) introduced history dependence to the rational expectations farsighted stability in Dutta and Vohra (2017) to assure nonemptiness of solutions for all finite problems.

[7] See Bogomolnaia and Jackson (2002) and Leo et al. (2021) as well.

2.1 One-to-One Matching Problems with Externalities

Let $M = \{m_1, ..., m_n\}$ and $W = \{w_1, ..., w_n\}$ be the sets of male and female agents with $|M| = |W| = n$. Let $\mu : M \cup W \to M \cup W$ be a one-to-one matching: $\mu(\mu(x)) = x$ for all $x \in M \cup W$ such that if $\mu(m) \neq m$ then $\mu(m) \in W$, and if $\mu(w) \neq w$ then $\mu(w) \in M$. The set of all matchings is denoted by $\mathcal{M}$. Each agent $x \in M \cup W$ has a complete, transitive, and reflexive preference relation R_x which is a binary relation over $\mathcal{M}$. Let the associated strict preference relation be $\mu P_x \mu'$ ($\mu R_x \mu'$ and $\neg \mu' R_x \mu$), and associated indifference relationship be $\mu I_x \mu'$ ($\mu R_x \mu'$ and $\mu' R_x \mu$). A matching μ is *fully matched* if $\mu(x) \neq x$ for all $x \in M \cup W$. Denote a set of all fully matched matchings by $\mathcal{M}^F$. A matching μ is a *fully unmatched matching* if $\mu(x) = x$ for all $x \in M \cup W$.

We define an effectiveness function which describes the resulting matching induced by a deviation from the original matching. The following effectiveness function is standard in the literature of matching theory and coalition formation (Roth and Vande Vate, 1990; Diamantoudi and Xue, 2003; Herings, Mauleon, and Vannetelboch, 2020).

Definition 1. *A matching μ'* ***is induced from*** *μ* ***by a pair*** $(m, w) \in M \times W$*, denoted by $\mu \to_{(m,w)} \mu'$, if it holds*

(i) $\mu(m) \neq w$ and $\mu'(m) = w$;

(ii) $\mu(m) \neq m \Rightarrow \mu'(\mu(m)) = \mu(m)$ and $\mu(w) \neq w \Rightarrow \mu'(\mu(w)) = \mu(w)$;

(iii) for all $x \in M \cup W \setminus \{m, w, \mu(m), \mu(w)\}$, $\mu(x) = \mu'(x)$.

In words, the effectiveness function states that when a pair of agents deviates from a matching, the resulting matching is identical to the original matching except that (1) deviators are matched, and (2) their previous partners are single. Similarly, we can define the effectiveness function for a deviation by an agent.

Definition 2. *A matching μ'* ***is induced from*** *μ* ***by an agent*** *$x \in M \cup W$, denoted by $\mu \rightarrow_{\{x\}} \mu'$, if it holds*

(i) $\mu(x) \neq x$ and $\mu'(x) = x$;

(ii) $\mu'(\mu(x)) = \mu(x)$;

(iii) for all $y \in M \cup W \setminus \{x, \mu(x)\}$, $\mu(y) = \mu'(y)$.

A matching μ is *pairwise stable* if for any $S \in (M \times W) \cup M \cup W$ and μ' with $\mu \rightarrow_S \mu'$, there exists $x \in S$ such that $\mu R_x \mu'$. We denote the set of pairwise stable matchings by $\mathcal{PS}$.

2.2 Pairs Competition Problems

In this section, we provide a tractable one-to-one matching problem with externalities.[8] Given a matching $\mu \in \mathcal{M}$, the pairs compete for a prize: both agents of the winning pair get a payoff of 1 each. An unmatched agent cannot participate in the contest and obtains a payoff of zero. Male and female agents m_i and w_i are characterized by their *abilities* a_{m_i} and a_{w_i}, respectively. We assume that $a_{m_1} > a_{m_2} > ... > a_{m_n}$ and $a_{w_1} > a_{w_2} > ... > a_{w_n}$. There are at most n pairs in the competition, and *we denote each pair by its male agent m_i's number $i = 1, ..., n$.* In this contest, each agent x of a pair chooses his/her effort level e_x simultaneously and independently. If m_i is matched under μ, pair i's members' efforts are aggregated by a CES function $Y_i = (a^{\sigma}_{m_i} e^{\sigma}_{m_i} + a^{\sigma}_{\mu(m_i)} e^{\sigma}_{\mu(m_i)})^{\frac{1}{\sigma}}$ with $\sigma \geq 0$.[9] If m_i is unmatched under μ, $Y_i = 0$. Given an aggregate effort profile $(Y_1, ..., Y_n)$, the winning probability for a pair is determined by a Tullock-style contest:

[8] This is a group contest game with endogenous group formation. Group formation in contests is first analyzed by Bloch, Sanchez-Pages, and Soubeyran (2006). Here, we consider a specific problem in which groups need to be pairs in two-sided matching setup. See Imamura, Konishi, and Pan (2021) for details.

[9] This CES aggregator function becomes a linear function (perfect substitutes) when $\sigma = 1$, and becomes a Cobb-Douglas function when $\sigma = 0$ in the limit.

pair i's *winning probability* π_i is given by

$$\pi_i = \frac{Y_i}{\sum_{k=1}^{n} Y_k}. \tag{1}$$

The effort cost function is common and linear for every agent x: $c_x(e_x) = e_x$. Therefore, the expected payoffs of agent x in pair i is

$$U_x = \pi_i - e_x + \epsilon a_{\mu(x)},$$

where $a_{\mu(x)}$ is agent x's payoff from the partner's ability, and $\epsilon > 0$ is sufficiently small. This ϵ is introduced to break ties when there is only one pair in the competition: agent x in the pair prefers a high ability partner even though he/she wins with probability one without making effort. Thus, in the pairs competition problem, for every agent x, preference P_x satisfies

(i) for all $\mu, \mu' \in \mathcal{M}$ with $\mu(x) \neq x$ and $\mu'(x) \neq x$, $\mu P_x \mu'$ or $\mu' P_x \mu$;

(ii) for all $\mu \in \mathcal{M}$ with $\mu(x) \neq x$ and $\mu' \in \mathcal{M}$ with $\mu'(x) = x$, $\mu P_x \mu'$;

(iii) for all $\mu, \mu' \in \mathcal{M}$ with $\mu(x), \mu'(x) \neq x$ and $a_{\mu(x)} > a_{\mu'(x)}$, $\mu P_x \mu'$;

(iv) for all $\mu, \mu' \in \mathcal{M}$ with $\mu(x) = \mu'(x) = x$, $\mu I_x \mu'$.

To provide further properties on preference P_x, we analyze equilibrium allocation of pairs competition problem under μ. We assume that pair i members regard the other groups' aggregate effort $Y_{-i} = \sum_{j \neq i} Y_j$ as given, and consider a Nash equilibrium of pair i's effort contribution game as the best response of pair i to the other pairs' aggregate effort Y_{-i}. Solving this problem, we obtain the total effort by all pairs

$$Y = \frac{n(\mu) - 1}{\sum_{j \in N(\mu)} \frac{1}{A_j(\mu)}},$$

where

- $N(\mu) \equiv \{i \in \{1, ..., n\} : \mu(m_i) \in W\}$ is the set of matched pairs under μ;

- $n(\mu) \equiv |N(\mu)|$ is the number of matched pairs under μ;
- $A_i(\mu) \equiv \left(a_{m_i}^{\frac{\sigma}{1-\sigma}} + a_{\mu(m_i)}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$ is the *productivity of pair* $i \in N(\mu)$.

Pair i's equilibrium winning probability is calculated as[10]

$$\pi_i = 1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n} \frac{1}{A_j(\mu)}}.$$

Member x of pair i's equilibrium payoff under μ when $\mu(x) \neq x$ can be explicitly solved as [11]

$$U_x = \underbrace{\left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j\in N(\mu)} \frac{1}{A_j(\mu)}}\right]}_{\text{winning probability}} \underbrace{\left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j\in N(\mu)} \frac{1}{A_j(\mu)}}\left(\frac{a_x}{A_i(\mu)}\right)\right]}_{\text{net benefits by taking effort dis utility into account}} + \epsilon a_{\mu(x)}.$$

Since agent x cannot control a_x, we can write U_x as:

$$U_x = V_x(A_i(\mu), E(\mu)) + \epsilon a_{\mu(x)},$$

where $E(\mu)$ is an aggregated externalities index under μ

$$E(\mu) \equiv \frac{\sum_{j\in N(\mu)} \frac{1}{A_j(\mu)}}{n(\mu)-1}.$$

Note that when agent x gets a higher ability partner, payoff U_x increases due to increases in both $A_i(\mu)$ and $E(\mu)$.

It is important to mention two properties of the aggregated externality index $E(\mu)$. First, $E(\mu)$ tends to decrease in the number of matched pair $n(\mu)$. This is because assuming that is the average value of $\frac{1}{A_j(\mu)}$, $\frac{1}{n(\mu)}\sum_{j\in N(\mu)} \frac{1}{A_j(\mu)}$, stays constant, $\frac{n(\mu)}{n(\mu)-1}$ decreases as $n(\mu)$ goes up. This externality causes an important difference between the standard matching problem and the one without externalities. The following example mentioned in the introduction illustrates that.

[10] For the detailed derivations, see Imamura, Konishi, and Pan (2021); Konishi, Pan, and Simeonov (2021).

[11] We can show that if $\sum_{j=1}^{n} \frac{1}{A_j(\mu)} > (n(\mu)-1)\frac{1}{A_i(\mu)}$ for all $i = 1, ..., n$, then every pair gets a positive winning probability, see Imamura, Konishi, Pan (2021); Konishi, Pan, and Simeonov (2021) for the details. This condition is satisfied for any $\mu \in \mathcal{M}$ if $\sum_{j=1}^{n} \frac{1}{A_j(\mu^*)} > (n-1)\frac{1}{A_i(\mu^*)}$ holds for the positive assortative matching μ^*.

Example 1. (Imamura, Konishi, and Pan, 2021) Consider a pairs competition problem with $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $a_{m_1} = a_{w_1} = 1$, $a_{m_2} = a_{w_2} = 0.9$, and $a_{m_3} = a_{w_3} = 0.7$. Set $\sigma = \frac{1}{2}$, then we have $Y_i = (a^{\frac{1}{2}}_{m_i} e^{\frac{1}{2}}_{m_i} + a^{\frac{1}{2}}_{\mu(m_i)} e^{\frac{1}{2}}_{\mu(m_i)})^2$ and $A_i = a_{m_i} + a_{\mu(m_i)}$. For simplicity set $\epsilon = 0$.[12] We calculate m_1's payoffs under the positive assortative matching μ^* and matching μ' with $\mu^* \rightarrow_{\{m_1, w_2\}} \mu'$.

(i) $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$:

$$U_{m_1}(\mu^*) = \left(1 - \frac{2 \times \frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}}\right)\left(1 - \frac{2 \times \frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}} \times \frac{1}{2}\right) = 0.312\,09$$

(ii) $\mu' = \{(m_1, w_2), (m_3, w_3)\}$:

$$U_{m_1}(\mu') = \left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.4}}\right)\left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.4}} \times \frac{1}{1.9}\right) = 0.447\,20$$

Thus, m_1 is better off by dumping his superior partner for an inferior partner. For any other fully matched matching $\mu \in M^F$, a similar deviation blocks μ. In addition, for any matching $\mu \notin M^F$, an unmatched pair blocks μ. Thus, there is no pairwise stable matching: i.e., $\mathcal{PS} = \emptyset$ in this example.■

Second, $E(\mu)$ is larger for more unequal ability distributions across pairs. This is because $E(\mu)$ is a increasing function in $\sum_{j \in N(\mu)} \frac{1}{A_j(\mu)}$. This means that if an assortative swapping occurs then the rest of active agents are all better off. This implies that if (m, w) deviates from a matching μ with $a_w > a_{\mu(m)}$ and $a_m > a_{\mu(w)}$, and if $\mu(m)$ and $\mu(w)$ form a pair to avoid being single (thus, an assortative swapping), then both higher ability agents m and w are better off, since such a swapping of partners to induce matching μ' satisfies $E(\mu) < E(\mu')$, $a_m + a_w > a_m + a_{\mu(m)}$, $a_m + a_w > a_{\mu(w)} + a_w$, $\frac{a_m}{a_m + a_w} < \frac{a_m}{a_m + a_{\mu(m)}}$, and $\frac{a_w}{a_m + a_w} < \frac{a_w}{a_{\mu(w)} + a_w}$. Imamura, Konishi, and Pan (2021) show the following lemma.

[12] The results derived from this example continue to hold for sufficiently small $\epsilon > 0$.

Lemma 1. *(Imamura, Konishi, and Pan, 2021) Let* μ, $m_\ell, m_k \in M$ *with* $\ell < k$ *(thus* $a_{m_\ell} > a_{m_k}$*), and* $\mu(m_\ell), \mu(m_k) \in W$ *with* $a(\mu(m_\ell)) < a(\mu(m_k))$. *Let* μ' *be such that* $\mu'(m_\ell) = \mu(m_k)$ *and* $\mu'(m_k) = \mu(m_\ell)$ *with* $\mu'(x) = \mu(x)$ *for all other* x *by swapping the partners among these two pairs. Then,* $E(\mu') > E(\mu)$ *holds.*

One important implication of Lemma 1 is that higher ability agents m_ℓ and $\mu(m_k)$ are better off by the above assortative swapping, since the abilities of their partners improve. We use these properties to analyze LCS in the next section.

3 The Results

3.1 LCS under the Standard Effectiveness Function

In this section, we consider farsighted agents and analyze the *largest consistent set* (LCS) introduced by Chwe (1994). We begin by providing a few concepts to define LCS.

Definition 3. *A matching* μ *is* ***indirectly dominated*** *by* μ' *if there is a finite sequence of distinct matchings* $\mu_0, ..., \mu_L$ *with* $\mu_0 = \mu$ *and* $\mu_L = \mu'$ *such that for every* $l \in \{0, ..., L-1\}$, $\mu_l \to_S \mu_{l+1}$ *holds for some* $S \in M \cup W \cup M \times W$ *such that* $\mu_L P_x \mu_l$ *for* $x \in S$. *We denote this indirect domination by* $\mu \ll \mu'$.

Definition 4. *A set of matchings* $\mathcal{CS}(\mathcal{M}) \subseteq \mathcal{M}$ *is* ***consistent*** *if for all* $\mu \in \mathcal{CS}(\mathcal{M})$, *all* μ' *induced by deviation* $\mu \to_S \mu'$ *for some* $S \in M \cup W \cup M \times W$, *there exist* $\tilde{\mu} \in \mathcal{CS}(\mathcal{M})$ *such that* $\mu' \ll \tilde{\mu}$, *and* $x \in S$ *with* $\neg \tilde{\mu} P_x \mu$.

Definition 5. *A set of matchings* $\mathcal{LCS}(\mathcal{M}) \subseteq \mathcal{M}$ *is the* ***largest consistent set*** *if it is consistent and contains all consistent set* $\mathcal{C}(\mathcal{M}) \subseteq \mathcal{LCS}(\mathcal{M})$.

Denote the positive assortative matching by μ^*, where $\mu^*(m_k) = w_k$ for all $k = 1, ..., n$. In pairs competition problems, μ^* satisfies the following property.

Lemma 2. *(1) For all $\mu \in \mathcal{M}^F$ with $\mu \neq \mu^*$, $\mu^* P_{m_1} \mu$ and $\mu^* P_{w_1} \mu$ hold, and (2) for all $k = 2, ..., n-1$, and all $\mu \in \mathcal{M}^F$ such that (i) $\mu(m_j) = w_j$ for all $j = 1, ..., k-1$, and (ii) $\mu(m_k) \neq w_k$, if $\mu \neq \mu^*$ then $\mu^* P_{m_k} \mu$ and $\mu^* P_{w_k} \mu$ hold.*

Proof. Suppose that $\mu \in \mathcal{M}^F$ and $\mu \neq \mu^*$. Then, there is k such that $\mu(m_k) \neq w_k$. Let the smallest of such k, and name it k. Then, $\mu(m_j) = w_j$ holds for all $j = 1, ..., k-1$, and $a_{\mu(m_k)} < a_{w_k}$ and $a_{\mu(w_k)} < a_{m_k}$. Consider a deviation by assortative swapping $\mu \rightrightarrows_{(m_k, w_k)} \mu'$. Since $\mu \in \mathcal{M}^F$, $\mu' \in \mathcal{M}^F$ holds. By Lemma 1, we have $\mu' P_{m_k} \mu$ and $\mu' P_{w_k} \mu$, and $\mu' P_{m_j} \mu$ and $\mu' P_{w_j} \mu$ for all $j = 1, ..., k-1$. Now, suppose that $\mu' \neq \mu^*$. By the same argument, there is the smallest $\ell > k$ with $\mu(m_\ell) \neq w_\ell$. Consider assortative swapping $\mu' \rightrightarrows_{(m_\ell, w_\ell)} \mu''$, then we have $\mu'' P_{m_k} \mu'$ and $\mu'' P_{w_k} \mu'$ by Lemma 1. Repeating this argument, we have $\mu^* P_{m_k} \mu$ and $\mu^* P_{w_k} \mu$. This proves (2). (1) can be shown similarly.■

This lemma shows that among matchings in $\mathcal{M}^F$, an under-externality version of the top coalition property introduced by Banerjee, Konishi, and Sönmez (2001) holds in the pairs competition problem. Diamantoudi and Xue (2003) show that the top-coalition property assures $\mathcal{LCS}(\mathcal{M}) = \{\mu^*\}$ without externalities. Even with externalities, we indeed have one direction of inclusion relationship.

Proposition 1. *In the pairs competition problem, $\mu^* \in \mathcal{LCS}(\mathcal{M})$ holds.*

Proof. It is easy to see that for all matching $\mu \in \mathcal{M}$, $\mu \ll \mu^*$ holds. In fact, consider deviation pairs (m_1, w_1), (m_2, w_2), ... , (m_n, w_n) in this order from μ, and let the first unmatched assortative pair deviate from μ. After the first deviation, let the next unmatched assotative pair deviate, and so on and so forth. This is an indirect dominance relationship, since μ^* is the best outcome for all assortative pairs, given that all higher ability assortative pairs have been matched by Lemma 1. This implies that a matching that results from a deviation from μ^* is indirectly dominated by μ^*. Thus, we conclude $\mu^* \in \mathcal{LCS}(\mathcal{M})$.□

Due to the assortative structure, one might think that LCS is a singleton of the assortative matching $\{\mu^*\}$. However, the other direction of inclusion relationship does not hold in the model with externalities: LCS includes not only μ^*, but also many other matchings. Perhaps surprisingly, LCS in Example 1 coincides with the set of all matchings $\mathcal{M}$, including the empty matching.

Proposition 2. *In Example 1,* $\mathcal{LCS}(\mathcal{M}) = \mathcal{M}$.

To prove the above statement, we introduce some notations. Let the sets of matchings with three, two, one, and zero pairs be $\mathcal{M}^3 = \{\mu \in \mathcal{M} : |\{x \in M \cup W : \mu(x) = x\}| = 0\}$, $\mathcal{M}^2 = \{\mu \in \mathcal{M} : |\{x \in M \cup W : \mu(x) = x\}| = 2\}$, $\mathcal{M}^1 = \{\mu \in \mathcal{M} : |\{x \in M \cup W : \mu(x) = x\}| = 4\}$, and $\mathcal{M}^0 = \{\mu \in \mathcal{M} : |\{x \in M \cup W : \mu(x) = x\}| = 6\}$, respectively.

For use later, we calculate m_1's payoffs under a few relevant matchings:

(i) $\mu_1 = \mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$:

$$U_{m_1}(\mu_1) = \left(1 - \frac{2 \times \frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}}\right)\left(1 - \frac{2 \times \frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}} \times \frac{1}{2}\right) = 0.312\,09$$

(ii) $\mu_2 = \{(m_1, w_3), (m_2, w_1)\}$:

$$U_{m_1}(\mu_2) = \left(1 - \frac{\frac{1}{1.7}}{\frac{1}{1.7} + \frac{1}{1.9}}\right)\left(1 - \frac{\frac{1}{1.7}}{\frac{1}{1.7} + \frac{1}{1.9}} \times \frac{1}{1.7}\right) = 0.32562$$

(iii) $\mu_3 = \{(m_1, w_1), (m_2, w_2)\}$:

$$U_{m_1}(\mu_3) = \left(1 - \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8}}\right)\left(1 - \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8}} \times \frac{1}{2}\right) = 0.40166$$

(iv) $\mu_4 = \{(m_1, w_2), (m_2, w_3)\}$:

$$U_{m_1}(\mu_4) = \left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.6}}\right)\left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.6}} \times \frac{1}{1.9}\right) = 0.41224$$

Naturally, we assume $U_x(\mu) = 0$ for all $x \in M \cup W$ if $\mu \in \mathcal{M}^0$. We also assume that if $\mu \in \mathcal{M}^1$, the pair wins with probability 1, but agents still slightly prefer a partner with higher ability: i.e., for x with $\mu(x) \neq x$, $U_x(\mu) = 1$ if $\mu(x) = m_1$ or $\mu(x) = w_1$, $U_x = 1 - \epsilon$ if $\mu(x) = m_2$ or $\mu(x) = w_2$, and $U_x = 1 - 2\epsilon$ if $\mu(x) = m_3$ or $\mu(x) = w_3$, where $\epsilon > 0$ is arbitrarily close to zero. This construction of payoffs of single pair matchings guarantees that for all $\mu \in \mathcal{M}^1$, all $\mu' \in \mathcal{M}^3 \cup \mathcal{M}^2$, and all x with $\mu(x) \neq x$ and $\mu'(x) \neq x$, $\mu P_x \mu'$ holds.

In this particular example, we can also show through direct calculation that for all $\mu \in \mathcal{M}^2$, all $\mu' \in \mathcal{M}^3$, and all x with $\mu(x) \neq x$ (and $\mu'(x) \neq x$), $\mu P_x \mu'$ holds. The calculations show that $U_{m_1}(\mu_1) < U_{m_1}(\mu_2)$ holds even though μ_1 is the most preferable matching in $\mathcal{M}^3$ for m_1 and μ_2 is the least preferable in $\mathcal{M}^2$ for m_1. We write down this property formally.

Strong Negative Externalities in Size (SNES). Suppose that (i) $\mu \in \mathcal{M}^1$ and $\mu' \in \mathcal{M}^2 \cup \mathcal{M}^3$, or (ii) $\mu \in \mathcal{M}^2$ and $\mu' \in \mathcal{M}^3$. If for all $x \in M \cup W$ with $\mu(x) \neq x$ and $\mu'(x) \neq x$, $\mu P_x \mu'$ holds.

With SNES, we can show the following claim.

Claim. In Example 1, $\mathcal{M}^1 \cup \mathcal{M}^2 \cup \mathcal{M}^3$ is consistent.

Proof. First pick $\mu \in \mathcal{M}^1$. Let $\mu(m) = w$. There are two potential deviation from μ. (Case 1) Suppose that a pair (m', w') with $m' \neq m$ and $w \neq w'$ deviates. The resulting matching μ' satisfies $\mu'(m) = w$ and $\mu'(m') = w'$. In this case, (m, w') can deviate from μ' creating $\mu''(m) = w'$. This is profitable by SNES. Thus, $\mu' \ll \mu''$ holds. Clearly, $\mu'' \in \mathcal{M}^1$ and $U_{m'}(\mu'') \not> U_{m'}(\mu)$. This shows that the pair (m', w') would not deviate from μ. (Case 2) Suppose that a pair (m, w') with $w' \neq w$ deviates from μ, creating μ' with $\mu'(m) = w'$. Consider $\mu'' \in \mathcal{M}^3$ with $\mu''(m) = w'$ by matching singles. Clearly $\mu' \ll \mu''$ holds and $U_m(\mu'') < U_m(\mu)$ by SNES. Again, (m, w') does not deviate from μ. (Possible deviations (m', w) also do not occur by symmetry.)

We can apply similar arguments to the case of $\mu \in \mathcal{M}^2$. If a deviation pair creates $\mu' \in \mathcal{M}^1$, then there is $\mu'' \in \mathcal{M}^3$ with $\mu' \ll \mu''$ by matching two single agents. By SNES, the original deviation is not profitable. If an agent deviates unilaterally by severing his/her match creating

$\mu' \in \mathcal{M}^1$, then there is $\mu'' \in \mathcal{M}^2$ with $\mu' \ll \mu''$ by matching a pair excluding the original deviator. Clearly, the original deviator does not benefit. If a deviation pair creates $\mu' \in \mathcal{M}^2$, then there is $\mu'' \in \mathcal{M}^3$ with $\mu' \ll \mu''$ by matching two single agents. By SNES, the original deviation is not profitable. If a deviation (m, w) creates $\mu' \in \mathcal{M}^3$ by matching two single agents, then w can deviate with $m' \neq m$ to create $\mu'' \in \mathcal{M}^2$ with $\mu' \ll \mu''$ by SNES. However, then $U_m(\mu'') \not> U_m(\mu)$ holds, and the original deviation is not profitable.

Finally, consider the case $\mu \in \mathcal{M}^3$. In this case, any deviation pair (m, w) generates $\mu' \in \mathcal{M}^2$. Let $\mu'(m) = w$ and $\mu'(m') = w'$ $(m' \neq m)$. In this case, suppose that (m', w) deviates from μ', creating $\mu'' \in \mathcal{M}^1$. By SNES, $\mu' \ll \mu''$ holds, and agent m is not better off. Even if a single's unilateral deviation creates $\mu' \in \mathcal{M}^2$, a further deviation from μ', creating $\mu'' \in \mathcal{M}^1$ does not make the original deviator better off. We have completed the proof.■

Proof of Proposition 2. In order to show that $\mathcal{M}$ is LCS, we need to show that the fully unmatched matching μ_0 ($\mu_0(x) = x$ for all $x \in M \cup W$) belongs to LCS. For this, we can utilize the above calculations. First, suppose that (m_1, w_1) deviates from μ_0. Then, (m_2, w_2) can deviate to create μ_3 (see above). However, then (m_1, w_2) and (m_2, w_3) sequentially deviate, creating μ_4. Agent m_1 is better off by moving from μ_3 to μ_4. Thus $\mu_3 \ll \mu_4$. However, the other initial deviator w_1 is not better off. Second, suppose that $(m, w) \neq (m_1, w_1)$ deviates. Since $(m, w) \neq (m_1, w_1)$, $m \neq m_1$ or $w \neq w_1$ holds. Without loss of generality, assume $m \neq m_1$. Then, after (m, w)'s deviation, (m_1, w) can deviate, and w is better off since we assumed that w slightly prefers a higher ability partner, and m cannot be better off by deviating with w initially. Hence, the LCS of this example is $\mathcal{M}$ itself.□

We conclude this section with a remark. The above result does not depend on pairwise deviations. Even if larger coalitions are allowed to deviate, both Propositions 1 and 2 continue to hold.

3.2 LCS under the Knuth Effectiveness Function

We consider the effectiveness function introduced by Knuth (1976) in this section. In our problem, unmatched agents get the lowest payoff of zero, since he/she cannot participate in the contest. Thus, it is not natural to think that a deviation's outcome leaves single agents who themselves can be paired. Since a deviation by a pair creates two single agents if both deviators had a partner, these abandoned agents might be considered to be the most natural ones to form a pair. We introduce a few concepts to formalize this idea.

Definition 6. *A matching $\mu' \in \mathcal{M}$ **is obtained from** $\mu \in \mathcal{M}$ **by swapping induced by a deviation** (m, w) if it holds*

(i) $\mu(m) \neq m$ and $\mu(w) \neq w$;

(ii) $\mu'(m) = w$ and $\mu'(\mu(w)) = \mu(m)$;

(iii) for all $x \in M \cup W \setminus \{m, w, \mu(m), \mu(w)\}$, $\mu(x) = \mu'(x)$.

Definition 7. ***Effectiveness function** $\rightrightarrows_S$ **on** $\mathcal{M}$ is such that (a) for $\mu \in \mathcal{M}$ and (m, w) with $\mu(m) \neq m$ and $\mu(w) \neq w$, $\mu \rightrightarrows_S \mu'$ iff μ' is obtained from μ by swapping induced by a deviation (m, w), and (b) for any other $\mu \in \mathcal{M}$ and $S \in M \cup W \cup M \times W$, $\mu \rightrightarrows_S \mu'$ iff $\mu \rightarrow_S \mu'$.*

Denote **LCS under the effectiveness function** $\rightrightarrows_S$ by $\mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$. Since under the effectiveness function with swapping $\rightrightarrows_S$, the number of pairs does not decrease as a result of a pairwise deviation. Imamura, Konishi, and Pan (2021) show that pairwise stable matching with the effectiveness function with swapping $\rightrightarrows_S$ exists in the pairs competition problems, which is the positive assortative matching μ^*. The reason is any anti-assortative deviation is not appealing to a higher ability agent, both because he/she will get an inferior partner and the externality index goes down.

Diamantoudi and Xue (2003) showed that if a hedonic game satisfies the top-coalition property, then LCS is the singleton core, which is the assortative matching in the one-to-one matching problem without externalities. Does the same result hold in our problem under the effectiveness function with swapping? The following proposition shows that the answer is affirmative.

Proposition 3. *In the pairs competition problem, LCS under effectiveness function $\rightrightarrows_S$ only includes μ^*:i.e., $\mathcal{LCS}_{\rightrightarrows}(\mathcal{M}) = \{\mu^*\}$.*

Proof. First notice $\mathcal{LCS}_{\rightrightarrows}(\mathcal{M}) \subseteq \mathcal{M}^F$. If μ has unmatched singles, any unmatched pair (m, w) can deviate from μ to obtain positive payoffs. Since both m and w will have partners under effectiveness function $\rightrightarrows_S$, after the deviation they retain positive payoffs, regardless of subsequent deviations. Since m and w obtain zero payoffs from matching μ, they certainly deviate from μ. Thus, $\mu \notin \mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$, and we conclude $\mathcal{LCS}_{\rightrightarrows}(\mathcal{M}) \subseteq \mathcal{M}^F$.

Now, we will prove $\mathcal{LCS}_{\rightrightarrows}(\mathcal{M}) = \{\mu^*\}$. First, we prove the following claim.

Claim. For all $\mu \in \mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$, we have $\mu(m_1) = w_1$.

Proof of Claim. Consider a set of full matchings in which m_1 and w_1 are not matched: $\mathcal{M}^F_{\neg 1} \equiv \{\mu \in \mathcal{M}^F : \mu(m_1) \neq w_1\}$. This is a finite set, and the elements of $\mathcal{M}^F_{\neg 1}$, $\mu^1, ..., \mu^K$ can be ordered by the values of their externality index in an increasing manner: $E(\mu^1) \leq E(\mu^2) \leq ... \leq E(\mu^K)$. We first show that $\mu^1 \notin \mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$. Consider $\mu^1 \rightrightarrows_{(m_1,w_1)} \tilde{\mu}^1$. Since $E(\tilde{\mu}^1) > E(\mu^1)$, neither m_1 nor w_1 has an incentive to perform an anti-assortative swapping from $\tilde{\mu}^1$ since $\tilde{\mu}^1 \not\ll \mu^1$. This proves $\mu^1 \notin \mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$. Second consider $\mu^2 \rightrightarrows_{(m_1,w_1)} \tilde{\mu}^2$. Since $E(\tilde{\mu}^2) > E(\mu^2)$, we have $\tilde{\mu}^2 \not\ll \mu^2$ and $\mu^2 \notin \mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$. Repeating the same argument, we conclude $\mu^k \notin \mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$ for all $k = 1, ..., K$. We completed the proof.■

We apply the above argument in the claim repeatedly. Let $\mathcal{M}^F_{\neg 2} \equiv \{\mu \in \mathcal{M}^F : \mu(m_1) = w_1$ and $\mu(m_2) \neq w_2\}$. This is a finite set, and its elements can be ordered by their externality index.

By the same argument, we conclude $\mathcal{M}^F_{\neg 2} \cap \mathcal{LCS}_{\rightrightarrows}(\mathcal{M}^F) = \emptyset$. So, we move on to $\mathcal{M}^F_{\neg 3} \equiv \{\mu \in \mathcal{M}^F : \mu(m_1) = w_1, \mu(m_2) = w_2 \text{ and } \mu(m_3) \neq w_3\}$, and so on. This proves that only μ^* remains in $\mathcal{LCS}_{\rightrightarrows}(\mathcal{M})$. We completed the proof.$\square$

4 Concluding Remarks

In this paper, we analyzed farsighted agents in a one-to-one matching problem with externalities and the assortative structure. In the matching problem without externalities of Becker (1973), the assortative matching is a quite robust prediction irrespective of pairwise or coalitional deviations and the choice of effectiveness function. However, with externalities, we showed that the choice of effectiveness function is crucial: LCS can be the set of all matchings with the Roth Vande Vate effectiveness function, or a singleton set of the assortative matching with the Knuth one. Thus, our results show that LCS is sensitive to the setup of the problem in the presence of externalities.

We conclude the paper with a couple of observations. First, in the definition of the consistent set, we assumed any matching in $\mathcal{M}$ can be in a consistent set. However, one may argue that if a matching is outside of $\mathcal{M}^F$, there is at least one eligible pair of male and female agents who are currently single. This means that they are getting the lowest possible payoffs, and they do not lose anything by trying to form a pair even though there may be a sequence of further deviations. If this argument is appealing, it might make sense to choose the candidates of consistent set from the subsets of $\mathcal{M}^F$. In the following, we will consider consistent sets in $\mathcal{M}^F$, while we use the standard effectiveness function and indirect domination relation $\ll$ defined on the set of all matchings $\mathcal{M}$. We can prove the following proposition by using almost the same argument as Proposition 3.

Proposition 4. *Consider subsets of $\mathcal{M}^F$ as consistent set candidates. Then, there is a unique consistent set $\{\mu^*\}$, which is $\mathcal{LCS}(\mathcal{M}^F)$.*

Second, we have focused on LCS to analyze farsighted agents. However, LCS is not the only solution concept for farsighted agents. The farsighted stable set—vNM stable set defined by indirect domination—have been extensively investigated in the recent literature. It is easy to see that the singleton set of the assortative matching $\{\mu^*\}$ is a farsighted stable set in our problem since μ^* indirectly dominates any other matchings. The question is whether or not this is the unique farsighted stable set in the pairs competition problem. Harsanyi's (1974) indirect domination requires every coalition participating in the chain reaction of proposals and counter-proposals to be better off (relative to their starting points) once the process terminates. If a coalition deviates from any matching in a consistent set, then there is a matching that indirectly dominates the matching generated by the deviation, and at least one of the coalition members weakly prefers the original matching to it. But indirect dominance does not require coalitions to choose their best moves, and it rules out possibly unwelcome interventions by other coalitions. This is a concern everywhere along the entire farsighted blocking chain, and Xue (1998), Ray and Vohra (2019), and Kimya (2020) among others analyzed farsighted stable sets defined various maximality conditions to restrict chains of coalitional deviations. Dutta and Vohra (2017) and Dutta and Vartiainen (2020) proposed rational expectations farsighted stable sets by assigning unique deviation move to each "state" without and with history-dependence, respectively.[13] These two papers also refine their solutions by using the idea of maximality. We leave further investigation of the farsighted stable sets for future research.

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